

UPR/0840-T
CTP TAMU-11/99
RU99-4-B
March 1999
hep-th/9903214

Embedding AdS Black Holes in Ten and Eleven Dimensions

M. Cvetič^{†1}, M.J. Duff^{‡2}, P. Hoxha[‡], James T. Liu^{*3}, H. Lü^{†1},
J.X. Lu^{‡2}, R. Martinez-Acosta[‡], C.N. Pope^{‡4}, H. Sati[‡] and T.A. Tran[‡]

[†]*Dept. of Phys. and Astro., University of Pennsylvania, Philadelphia, PA 19104*

[‡]*Center for Theoretical Physics, Texas A&M University, College Station, TX 77843*

^{*}*Dept. of Phys., The Rockefeller University, New York, NY 10021*

ABSTRACT

We construct the non-linear Kaluza-Klein ansätze describing the embeddings of the $U(1)^3$, $U(1)^4$ and $U(1)^2$ truncations of $D = 5$, $D = 4$ and $D = 7$ gauged supergravities into the type IIB string and M-theory. These enable one to oxidise any associated lower dimensional solutions to $D = 10$ or $D = 11$. In particular, we use these general ansätze to embed the charged AdS_5 , AdS_4 and AdS_7 black hole solutions in ten and eleven dimensions. The charges for the black holes with toroidal horizons may be interpreted as the angular momenta of D3-branes, M2-branes and M5-branes spinning in the transverse dimensions, in their near-horizon decoupling limits. The horizons of the black holes coincide with the worldvolumes of the branes. The Kaluza-Klein ansätze also allow the black holes with spherical or hyperbolic horizons to be reinterpreted in $D = 10$ or $D = 11$.

¹ Research supported in part by DOE grant DOE-FG02-95ER40893

² Research supported in part by NSF Grant PHY-9722090

³ Research supported in part by DOE grant DOE-91ER40651-TASKB

⁴ Research supported in part by DOE grant DOE-FG03-95ER40917

Contents

1	Introduction	2
2	S^5 reduction of type IIB supergravity	5
2.1	Reduction ansätze	5
2.2	$D = 5$ AdS black holes	7
2.3	Rotating D3-brane	7
3	S^7 reduction of $D = 11$ supergravity	10
3.1	Reduction ansätze	10
3.2	$D = 4$ AdS black holes	13
3.3	Rotating M2-brane	14
4	S^4 reduction of $D = 11$ supergravity	15
4.1	Reduction ansätze	15
4.2	$D = 7$ AdS black holes	17
4.3	Rotating M5-brane	17
5	Sphere reduction of generic rotating p-branes, and domain wall black holes	19
6	Conclusions	20
A	Single-charge rotating p-branes	22
B	$D = 4$, $N = 2$ gauged supergravity	23
C	Calculation of the Ricci tensor for the reduction ansätze	27

1 Introduction

Anti-de Sitter black hole solutions of gauged extended supergravities [1] are currently attracting a good deal of attention [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] due, in large part, to the correspondence between anti-de Sitter space and conformal field theories on its boundary [13, 14, 15, 16]. These gauged extended supergravities can arise as the massless modes of various Kaluza-Klein compactifications of both $D = 11$ and $D = 10$ supergravities. The three examples studied in the paper will be gauged $D = 4$, $N = 8$ $SO(8)$ supergravity [17, 18] arising from $D = 11$ supergravity on S^7 [19, 20] whose black hole solutions are discussed in [7]; gauged $D = 5$, $N = 8$ $SO(6)$ supergravity [21, 22] arising from Type IIB supergravity on S^5 [23, 24, 25] whose black hole solutions are discussed in [2, 6]; and gauged $D = 7$, $N = 4$ $SO(5)$ supergravity [21, 26] arising from $D = 11$ supergravity on S^4 [27] whose black hole solutions are given in section 4.2 and in [9, 28].¹ In the absence of the black holes, these three AdS compactifications are singled out as arising from the near-horizon geometry of the extremal non-rotating M2, D3 and M5 branes [29, 30, 31, 32]. One of our goals will be to embed these known lower-dimensional black hole solutions into ten or eleven dimensions, thus allowing a higher dimensional interpretation in terms of rotating M2, D3 and M5-branes.

Since these gauged supergravity theories may be obtained by consistently truncating the massive modes of the full Kaluza-Klein theories, it follows that all solutions of the lower-dimensional theories will also be solutions of the higher-dimensional ones [33, 34]. In principle, therefore, once we know the Kaluza-Klein ansatz for the massless sector, it ought to be straightforward to read off the higher dimensional solutions. In practice, however, this is a formidable task. The correct massless ansatz for the S^7 compactification took many years to finalize [35, 36], and is still highly implicit, while for the S^5 and S^4 compactifications, the complete massless ansätze are still unknown. For our present purposes, it suffices to consider truncations of the gauged supergravities to include only gauge fields in the Cartan subalgebras of the full gauge groups, namely $U(1)^4$, $U(1)^3$ and $U(1)^2$ for the S^7 , S^5 and S^4 compactifications, respectively. These truncated theories will admit respectively the 4-charge AdS_4 , 3-charge AdS_5 and 2-charge AdS_7 black hole solutions.

The simplest of the three is perhaps the $D = 5$, $N = 8$ maximal gauged supergravity, for which there is a consistent $N = 2$ (*i.e.* minimal) truncation to supergravity coupled to

¹BPS black holes arising in the $SU(2) \times SU(2)$ version of gauged $N = 4$ supergravity in $D = 4$, which is the massless sector of the $S^3 \times S^3$ compactification of $N = 1$ supergravity in $D = 10$, were discussed in [5]. These solutions are not asymptotically AdS.

two abelian vector multiplets. This has the bosonic field content of a graviton, three $U(1)$ gauge fields and two scalars. In this paper we obtain the complete non-linear Kaluza-Klein ansatz for the compactification of $D = 10$ Type IIB supergravity on S^5 , truncated to the $U(1)^3$ Cartan subgroup of $SO(6)$.

In four dimensions there is a consistent truncation of gauged $N = 8$ maximal supergravity to gauged $N = 2$ supergravity coupled to three vector multiplets. The bosonic sector consists of a graviton, four vectors and three complex scalars, whose real and imaginary parts correspond to three “axions” and three “dilaton.”² The inclusion of the axions is necessary for providing a consistent truncation; the full bosonic Lagrangian in this case is obtained in appendix B. This truncation corresponds to the $U(1)^4$ Cartan subgroup of the non-abelian $SO(8)$, for which there exist AdS black hole solutions with four electric charges [7]. While one would ideally wish to obtain a complete Kaluza-Klein ansatz for the $N = 2$ truncation, in practice the complexity arising from the inclusion of the axions is considerable. Thus in the present paper we omit the axions in the Kaluza-Klein reduction. This is of course sufficient for the embedding of the electric black hole solutions in $D = 11$ as they do not involve the axions.

Finally, in seven dimensions, maximal $N = 4$ gauged $SO(5)$ supergravity admits a consistent truncation to $N = 2$ supergravity, comprising the metric, a 2-form potential, three vectors and a dilaton, coupled to a vector multiplet comprising a vector and three scalars. We obtain the Kaluza-Klein ansatz for an S^4 reduction of $D = 11$ supergravity, including two $U(1)^2$ gauge fields and two dilatonic scalars. This is sufficient for the consideration of the embedding of the $D = 7$ black holes in $D = 11$.

Having obtained the explicit Kaluza-Klein reduction ansätze, this allows an investigation of the embedding of the various AdS black holes of $D = 4$, $D = 5$ and $D = 7$ in the respective higher-dimensional supergravities. An important point here is that one must know the exact Kaluza-Klein reduction ansatz for the reduction of the supergravity theory itself, and not just for a specific solution, in order to show that the metric, gauge fields and scalar fields of the lower-dimensional solution are indeed precisely embeddable in the higher-dimensional

²Interestingly enough, the ungauged version of this theory obtained by switching off the gauge coupling and performing some dualisations, appears in the T^2 compactification of $D = 6$, $N = 1$ string theory. The four vectors are the two Kaluza-Klein and two winding gauge fields, while the three complex scalars S , T and U correspond to the axion-dilaton, the Kahler form and complex structure of the torus. This STU system plays a crucial role in four-dimensional string/string/string triality [37]. The black hole solutions of this theory [38, 37], and their embedding in ungauged $N = 8$ supergravity [39, 40] arising from the T^7 compactification of M -theory as intersections [41, 42] are also well known.

theory. It is worth remarking, in this regard, that it is the scalar fields that present most of the subtleties and complexities in the implementation of the reduction procedure.

Having embedded these black holes in ten or eleven dimensions, an interesting question then arises as to their higher-dimensional interpretation. It was noted some time ago [43], in the context of a “test” membrane moving in a fixed $\text{AdS}_4 \times S^7$, that a 4-dimensional BPS state (whose AdS energy is equal to its electric charge) admits the eleven-dimensional interpretation of an M2-brane [44, 45] that is rotating in the extra dimensions. Moreover, the electric charge is equal to the spin.

Recently there has been an upsurge of activities on the study of rotating p -branes [46, 47, 48, 49, 50, 51, 8, 9, 10, 52, 53, 54, 55]. In particular, in [8] AdS Reissner-Nordström black holes (*i.e.* the charged black holes without scalars) of AdS supergravity in $D = 4$ and $D = 5$ were studied, and shown to be related to the rotating solutions of M-/string theory. In [9] the near-extreme spinning D3-brane with one angular momentum was shown to reproduce the metric and the gauge fields of the large ($k = 0^+$ limit) of $D = 5$ gauged supergravity black holes [6], with the anticipation that the result would generalise to multiple angular momenta. However, the identification of the scalar fields was not given. (In addition, in [9, 10], the equivalence of the thermodynamics of the near-extreme spinning branes and the corresponding large black holes of $D = 4, 5, 7$ gauged supergravity was given.) While incomplete, these works provided some initial stages in the investigation of the sphere compactifications of M-/string theory.

Unlike black holes that are asymptotically Minkowskian, for which the horizons are always spherical, it is known that AdS black holes can also admit horizons of more general topology. Following the embedding procedure described above, we demonstrate that AdS_4 black hole with toroidal horizon can indeed be interpreted as the near-horizon structures of an M2-brane rotating in the extra dimensions. The four charges corresponding to the $U(1)^4$ Cartan subgroup are just the four angular momenta. Similarly, the 5-dimensional charged black hole with toroidal horizon corresponds to a rotating D3-brane and the 7-dimensional charged black hole with toroidal horizon to a rotating M5-brane. In each case, the event horizon coincides with the worldvolume of the brane.³ Additionally, one may use the Kaluza-Klein ansatz to obtain the higher-dimensional interpretation of AdS black holes with horizons of other topologies. We conjecture that these correspond to the near-horizon limits of rotating p -branes whose world-volumes have these topologies. (In fact the rotating “test” membrane in [43] had S^2 topology.)

³This is a concrete realisation of the “Membrane Paradigm” [56].

In this paper we also obtain the general rotating p -brane solutions in arbitrary dimensions, supported by a single $(p+2)$ -form charge, and discuss their sphere reductions. These rotating p -branes are easily constructed, merely by performing standard diagonal dimensional oxidations of the general rotating black holes that were constructed in [46].

2 S^5 reduction of type IIB supergravity

The S^5 reduction [23, 24, 25] of type IIB supergravity gives rise to $N = 8$, $D = 5$ gauged supergravity, with $SO(6)$ Yang-Mills gauge group [21, 22]. The complete details of this reduction, as with any sphere reduction, would be of great complexity, and in fact no example has ever been fully worked out. For our present purposes, however, it suffices to consider the truncation of the five-dimensional theory to $N = 2$ supersymmetry. In this truncation, which is of course a consistent one, the gauge group is reduced down to the $U(1) \times U(1) \times U(1)$ Cartan subgroup of $SO(6)$. The bosonic sector of the theory comprises these three gauge bosons, the metric, and two scalar fields. (The consistency of the truncation to this field content can be seen by considering the S^1 reduction of ungauged minimal non-chiral supergravity in $D = 6$, whose bosonic fields $(g_{\mu\nu}, A_{(2)}, \phi)$ reduce to give precisely the field content we are considering here in $D = 5$. After gauging, one would obtain the $U(1)^3$ gauged theory.)

2.1 Reduction ansätze

Even to construct the S^5 reduction ansatz for this truncated $N = 2$ theory is somewhat non-trivial, owing to the presence of the scalar fields. It is most conveniently expressed in terms of the parameterisation of sphere metrics given in [57].

We find that the ansatz for the reduction of the ten-dimensional metric is

$$ds_{10}^2 = \sqrt{\tilde{\Delta}} ds_5^2 + \frac{1}{g^2 \sqrt{\tilde{\Delta}}} \sum_{i=1}^3 X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + g A^i)^2 \right), \quad (2.1)$$

where the two scalars are parameterised in terms of the three quantities X_i , which are subject to the constraint $X_1 X_2 X_3 = 1$. They can be parameterised in terms of two dilatons φ_1 and φ_2 as

$$X_i = e^{-\frac{1}{2} \vec{a}_i \cdot \vec{\varphi}}, \quad (2.2)$$

where \vec{a}_i satisfy the dot products

$$M_{ij} \equiv \vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - \frac{4}{3}. \quad (2.3)$$

A convenient choice is

$$\vec{a}_1 = (\frac{2}{\sqrt{6}}, \sqrt{2}) , \quad \vec{a}_2 = (\frac{2}{\sqrt{6}}, -\sqrt{2}) , \quad \vec{a}_3 = (-\frac{4}{\sqrt{6}}, 0) . \quad (2.4)$$

The three quantities μ_i are subject to the constraint $\sum_i \mu_i^2 = 1$, and the metric on the unit round 5-sphere can be written in terms of these as

$$d\Omega_5^2 = \sum_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) . \quad (2.5)$$

The μ_i can be parameterised in terms of angles on a 2-sphere, for example as

$$\mu_1 = \sin \theta , \quad \mu_2 = \cos \theta \sin \psi , \quad \mu_3 = \cos \theta \cos \psi . \quad (2.6)$$

Note that $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \sum_{i=1}^3 X_i \mu_i^2 , \quad (2.7)$$

and is therefore expressed purely in terms of the scalar fields, and the coordinates on the compactifying 5-sphere. The constant g in (2.1) is the inverse of the radius of the compactifying 5-sphere, and is equal to the gauge coupling constant. We find that the ansatz for the reduction of the 5-form field strength is $F_{(5)} = G_{(5)} + *G_{(5)}$, where

$$\begin{aligned} G_{(5)} = & 2g \sum_i \left(X_i^2 \mu_i^2 - \tilde{\Delta} X_i \right) \epsilon_{(5)} - \frac{1}{2g} \sum_i X_i^{-1} \bar{*} dX_i \wedge d(\mu_i^2) \\ & + \frac{1}{2g^2} \sum_i X_i^{-2} d(\mu_i^2) \wedge (d\phi_i + g A_{(1)}^i) \wedge \bar{*} F_{(2)}^i . \end{aligned} \quad (2.8)$$

Here, $F_{(2)}^i = dA_{(1)}^i$, $\epsilon_{(5)}$ is the volume form of the 5-dimensional metric ds_5^2 , and $\bar{*}$ denotes the Hodge dual with respect to the five-dimensional metric ds_5^2 .

Substituting these ansätze into the equations for motion for the type IIB theory, we obtain five-dimensional equations of motion that can be derived from the Lagrangian⁴

$$e^{-1} \mathcal{L}_5 = R - \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}(\partial\varphi_2)^2 + 4g^2 \sum_i X_i^{-1} - \frac{1}{4} \sum_i X_i^{-2} (F_{(2)}^i)^2 + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^1 F_{\rho\sigma}^2 A_\lambda^3 . \quad (2.9)$$

(The other bosonic fields of the type IIB theory are set to zero in this $U(1)^3$ truncated reduction.) Note that the ten-dimensional Bianchi identity $dF_{(5)} = 0$ gives rise to the equations of motion for the scalars and gauge fields in five dimensions.

Thus we have established that the reduction ansätze (2.1) and (2.8) describe the exact embedding of the five-dimensional $N = 2$ gauged $U(1)^3$ supergravity into type IIB supergravity.

⁴We shall make some more detailed comments on certain general features of these spherical Kaluza-Klein reductions in section 3, where we consider the S^7 reduction of $D = 11$ supergravity.

The bosonic Lagrangian (2.9) can be further truncated down to smaller sectors. For example, we can consistently set $\varphi_2 = 0$, implying that $X_1 = X_2 = X_3^{-1/2}$, provided that $F_{(2)}^1 = F_{(2)}^2 = F_{(2)}/\sqrt{2}$. The Lagrangian then becomes

$$e^{-1}\mathcal{L}_5 = R - \frac{1}{2}(\partial\varphi_1)^2 + 4g^2(2e^{\frac{1}{\sqrt{6}}\varphi_1} + e^{-\frac{2}{\sqrt{6}}\varphi_1}) - \frac{1}{4}e^{\frac{2}{\sqrt{6}}\varphi_1}(F_{(2)})^2 - \frac{1}{4}e^{-\frac{4}{\sqrt{6}}\varphi_1}(F_{(2)}^3)^2 + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda^3. \quad (2.10)$$

It is also possible to set both scalars to zero, implying that $X_i = 1$, provided that $F_{(2)}^i = F_{(2)}/\sqrt{3}$. The Lagrangian is then given by

$$e^{-1}\mathcal{L}_5 = R + 12g^2 - \frac{1}{4}F_{(2)}^2 + \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda. \quad (2.11)$$

The embedding of the truncated Lagrangian (2.11) in $D = 10$ dimensions was discussed in [8].

2.2 $D = 5$ AdS black holes

The Lagrangian (2.9) admits a three-charge AdS black hole solution, given by [6]

$$ds_5^2 = -(H_1H_2H_3)^{-2/3}f dt^2 + (H_1H_2H_3)^{1/3}(f^{-1}dr^2 + r^2d\Omega_{3,k}^2), \\ X_i = H_i^{-1}(H_1H_2H_3)^{1/3}, \quad A_{(1)}^i = \sqrt{k}(1 - H_i^{-1})\coth\beta_i dt, \quad (2.12)$$

and

$$f = k - \frac{\mu}{r^2} + g^2 r^2 (H_1H_2H_3), \quad H_i = 1 + \frac{\mu \sinh^2 \beta_i}{k r^2}. \quad (2.13)$$

Here k can be 1, 0 or -1 , corresponding to the foliating surfaces of the transverse space being S^3 , T^3 or H^3 , with unit metric $d\Omega_{3,k}^2$, where H^3 denotes the hyperbolic 3-space of constant negative curvature. In the case of $k = 0$, one first needs to make the rescaling [9] $\sinh^2 \beta_i \rightarrow k \sinh^2 \beta_i$, followed by sending k to zero. The gauge potential for $k = 0$ case is then given by

$$A_{(1)}^i = \frac{1 - H_i^{-1}}{\sinh \beta_i} dt. \quad (2.14)$$

2.3 Rotating D3-brane

In this section, we show that the $k = 0$ three-charge AdS black hole of the $N = 2$ gauged supergravity in $D = 5$ given in (2.12) can be embedded in $D = 10$ as a solution that is precisely the decoupling limit of the rotating D3-brane. The higher-dimensional solutions corresponding to five-dimensional AdS black holes with $k = 1$ and $k = -1$ can also be easily obtained, by substituting the five-dimensional solutions into the S^5 reduction ansätze.

There can be three angular momenta, ℓ_i , $i = 1, 2, 3$, in the rotating D3-brane. The generic single-charge rotating p -branes, which can be obtained by dimensional oxidation of the generic single-charge rotating black holes constructed in [46], are presented in appendix A. We find that the metric of the rotating D3-brane is given by⁵

$$\begin{aligned}
ds_{10}^2 = & H^{-\frac{1}{2}} \left(- \left(1 - \frac{2m}{r^4 \Delta} \right) dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + H^{\frac{1}{2}} \left[\frac{\Delta dr^2}{H_1 H_2 H_3 - 2m r^{-4}} \right. \\
& + r^2 \sum_{i=1}^3 H_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) - \frac{4m \cosh \alpha}{r^4 H \Delta} dt \left(\sum_{i=1}^3 \ell_i \mu_i^2 d\phi_i \right) \\
& \left. + \frac{2m}{r^4 H \Delta} \left(\sum_{i=1}^3 \ell_i \mu_i^2 d\phi_i \right)^2 \right], \tag{2.15}
\end{aligned}$$

where the functions Δ , H , and H_i are given by

$$\begin{aligned}
\Delta &= H_1 H_2 H_3 \sum_{i=1}^3 \frac{\mu_i^2}{H_i}, \quad H = 1 + \frac{2m \sinh^2 \alpha}{r^4 \Delta}, \\
H_i &= 1 + \frac{\ell_i^2}{r^2}, \quad i = 1, 2, 3. \tag{2.16}
\end{aligned}$$

The rotating D3-brane is supported by the self-dual 5-form field strength $F_{(5)}$ of the type IIB theory. It is given by $F_{(5)} = G_{(5)} + *G_{(5)}$, where $G_{(5)} = dB_{(4)}$ and

$$B_{(4)} = \frac{1 - H^{-1}}{\sinh \alpha} \left(- \cosh \alpha dt + \sum_{i=1}^3 \ell_i \mu_i^2 d\phi_i \right) \wedge d^3 x. \tag{2.17}$$

As is well known, the non-rotating D3-brane has a “decoupling limit” where the space-time of the D3-brane becomes a product space $M_5 \times S^5$. If the D3-brane is extremal, M_5 is a five-dimensional anti-de Sitter spacetime. More generally, when the D3-brane is non-extremal, M_5 is the Carter-Novotny-Horsky metric [58], which can thus be viewed as a “non-extremal” generalisation of AdS_5 . A similar limit also exists for the rotating D3-brane, and can be achieved by making the rescalings

$$\begin{aligned}
m &\longrightarrow \epsilon^4 m, \quad \sinh \alpha \longrightarrow \epsilon^{-2} \sinh \alpha, \\
r &\longrightarrow \epsilon r, \quad x^\mu \longrightarrow \epsilon^{-1} x^\mu, \quad \ell_i \longrightarrow \epsilon \ell_i, \tag{2.18}
\end{aligned}$$

and then sending $\epsilon \longrightarrow 0$. (Note that when this limit is taken, we also have $\cosh \alpha \longrightarrow \epsilon^{-2} \sinh \alpha$.) This has the effect that the last term in (2.15) is set to zero and that

$$H = 1 + \frac{2m \sinh^2 \alpha}{r^4 \Delta} \longrightarrow \frac{2m \sinh^2 \alpha}{r^4 \Delta}. \tag{2.19}$$

⁵This metric agrees with previously obtained results [52, 53], after correcting some typographical errors.

In this limit, the metric (2.15) becomes

$$ds_{10}^2 = \sqrt{\tilde{\Delta}} \left[- (H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} (f^{-1} dr^2 + r^2 d\vec{y} \cdot d\vec{y}) \right] + \frac{1}{g^2 \sqrt{\tilde{\Delta}}} \sum_{i=1}^3 X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + g A^i)^2 \right), \quad (2.20)$$

where

$$\vec{y} = g \vec{x}, \quad g^2 = \frac{1}{\sqrt{2m} \sinh \alpha}, \quad \mu = 2m g^2. \quad (2.21)$$

The metric (2.20) precisely matches the dimensional reduction ansatz (2.1), with the lower dimensional fields given by

$$ds_5^2 = - (H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} (f^{-1} dr^2 + r^2 d\vec{y} \cdot d\vec{y}), \\ X_i = H_i^{-1} (H_1 H_2 H_3)^{1/3}, \quad A_{(1)}^i = \frac{1 - H_i^{-1}}{g \ell_i \sinh \alpha} dt, \quad (2.22)$$

where

$$f = -\frac{\mu}{r^2} + g^2 r^2 H_1 H_2 H_3, \quad g^2 = \frac{1}{\sqrt{2m} \sinh \alpha}, \quad \mu = 2m g^2. \quad (2.23)$$

To complete the story, we note that the 5-form field strength in the decoupling limit is given by $F_{(5)} = G_{(5)} + *G_{(5)}$, where $G_{(5)} = dB_{(4)}$ and

$$B_{(4)} = -g^4 r^4 \Delta dt \wedge d^3 x + \frac{1}{\sinh \alpha} \left(\sum_{i=1}^3 \ell_i \mu_i^2 d\phi_i \right) \wedge d^3 x. \quad (2.24)$$

This gives precisely the field strength in the dimensional reduction ansatz (2.8).

Thus we see that the solution (2.22) is precisely the $k = 0$ three-charge AdS black hole given in the previous subsection, after reparameterising the angular momenta $\ell_i^2 = \mu \sinh^2 \beta_i$. This shows that the embedding of the three-charge AdS $k = 0$ black hole in gauged $N = 2$ supergravity in five dimensions gives a ten-dimensional solution that is precisely the decoupling limit of the rotating D3-brane. Single-charge AdS black holes coming from the reduction of the metric of a rotating D3-brane with one angular momentum was obtained in [9], however without the explicit embedding of the scalar fields. The connection between the thermodynamics of AdS black holes and rotating p -branes was discussed in [9, 10].

It is also straightforward to oxidise the $k = 1$ and $k = -1$ AdS black holes back to $D = 10$ type IIB. The metric is the same form as (2.20) with $d\vec{y} \cdot d\vec{y}$ replaced by the unit metric for S^3 or H^3 respectively. The 5-form field strength follows by substituting the five-dimensional fields into (2.8).

3 S^7 reduction of $D = 11$ supergravity

3.1 Reduction ansätze

The S^7 reduction of eleven-dimensional supergravity gives rise to $SO(8)$ gauged $N = 8$ supergravity in four dimensions. One may again consider a consistent truncation to $N = 2$, for which the bosonic sector comprises the metric, four commuting $U(1)$ gauge potentials, three dilatons and three axions. (That this is a consistent truncation can be seen by reducing minimal non-chiral six-dimensional supergravity on T^2 , for which the reduction of $(g_{\mu\nu}, A_{(2)}, \phi)$ will give precisely the field content we are considering. After gauging, this would give the $U(1)^4$ gauged theory. See appendix B for an extended discussion of this.) We have not yet determined the complete reduction ansatz for the entire truncated theory where the axions are included, but we can give the exact ansatz in the case where one sets the axions to zero. This will not, of course, be a consistent truncation, since the $U(1)$ gauge fields will provide source terms of the form $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ for the axions. Nevertheless, one can use the axion-free ansatz for discussing the exact embedding of four-dimensional solutions for which the axions are zero. The full $N = 2$ four-dimensional theory, including the axions, is obtained in appendix B.

The reduction ansatz for the eleven-dimensional metric is

$$ds_{11}^2 = \tilde{\Delta}^{2/3} ds_4^2 + g^{-2} \tilde{\Delta}^{-1/3} \sum_i X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2 \right). \quad (3.1)$$

where $\tilde{\Delta} = \sum_{i=1}^4 X_i \mu_i^2$. The four quantities μ_i satisfy $\sum_i \mu_i^2 = 1$. They can be parameterised in terms of angles on the 3-sphere as

$$\mu_1 = \sin \theta, \quad \mu_2 = \cos \theta \sin \varphi, \quad \mu_3 = \cos \theta \cos \varphi \sin \psi, \quad \mu_4 = \cos \theta \cos \varphi \cos \psi. \quad (3.2)$$

The four X_i , which satisfy $X_1 X_2 X_3 X_4 = 1$, can be parameterised in terms of three dilatonic scalars $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$:

$$X_i = e^{-\frac{1}{2} \vec{a}_i \cdot \vec{\varphi}}, \quad (3.3)$$

where the \vec{a}_i satisfy the dot products

$$M_{ij} \equiv \vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - 1. \quad (3.4)$$

A convenient choice, corresponding to the combinations of (B.11), is

$$\vec{a}_1 = (1, 1, 1), \quad \vec{a}_2 = (1, -1, -1), \quad \vec{a}_3 = (-1, 1, -1), \quad \vec{a}_4 = (-1, -1, 1). \quad (3.5)$$

The reduction ansatz for the 4-form field strength is

$$\begin{aligned}
F_{(4)} &= 2g \sum_i \left(X_i^2 \mu_i^2 - \tilde{\Delta} X_i \right) \epsilon_{(4)} + \frac{1}{2g} \sum_i X_i^{-1} \bar{*} dX_i \wedge d(\mu_i^2) \\
&\quad - \frac{1}{2g^2} \sum_i X_i^{-2} d(\mu_i^2) \wedge (d\phi_i + g A_{(1)}^i) \wedge \bar{*} F_{(2)}^i .
\end{aligned} \tag{3.6}$$

Here, $\bar{*}$ denotes the Hodge dual with respect to the four-dimensional metric ds_4^2 , and $\epsilon_{(4)}$ denotes its volume form.⁶

It is of interest to note that the eleven-dimensional Bianchi identity $dF_{(4)} = 0$ already gives rise to the four-dimensional equations of motion for the scalars and gauge potentials, namely

$$\begin{aligned}
d\bar{*}d\log(X_i) &= \frac{1}{4} \sum_j M_{ij} X_j^{-2} \bar{*} F_{(2)}^j \wedge F_{(2)}^j + g^2 \sum_{j,k} M_{ij} X_j X_k - g^2 \sum_j M_{ij} X_j^2 , \\
d(X_i^{-2} \bar{*} F_{(2)}^i) &= 0 .
\end{aligned} \tag{3.7}$$

It is straightforward to see that these equations of motion can be obtained from the four-dimensional Lagrangian

$$e^{-1} \mathcal{L}_4 = R - \frac{1}{2} (\partial \vec{\varphi})^2 + 8g^2 (\cosh \varphi_1 + \cosh \varphi_2 + \cosh \varphi_3) - \frac{1}{4} \sum_{i=1}^4 e^{\vec{a}_i \cdot \vec{\varphi}} (F_{(2)}^i)^2 . \tag{3.8}$$

One might think that it would be possible to obtain the four-dimensional Lagrangian by substituting the ansätze (3.1) and (3.6) into the eleven-dimensional Lagrangian. In fact this is not the case, and one must work at the level of the eleven-dimensional equations of motion. One way of understanding this is from the fact that the ansatz for $F_{(4)}$ does not *identically* satisfy the Bianchi identity. Rather, as we have seen, it satisfies it modulo the use of the four-dimensional equations of motion for the scalars and gauge fields. In other words, the ansatz is made on the eleven-dimensional 4-form $F_{(4)}$ rather than on the fundamental potential $A_{(3)}$ itself. Consequently, it would not be correct to insert the ansatz for $F_{(4)}$ into the Lagrangian.

We may further illustrate this point by showing, as an example, how the scalar potential arises in the four-dimensional Einstein equation. This comes from considering the eleven-dimensional Einstein equation,

$$\hat{R}_{AB} - \frac{1}{2} \hat{R} \eta_{AB} = \frac{1}{12} \left(F_{AB}^2 - \frac{1}{8} F^2 \eta_{AB} \right) . \tag{3.9}$$

⁶If the ansätze (3.1) and (3.6) are linearised around an $\text{AdS}_4 \times S^7$ background, they can be seen to be in agreement with previous results that were derived at the linear level [19]. The full non-linear metric ansatz (3.1) should be in agreement with the appropriate specialisation of the ansatz given in [59].

with vielbein indices A, B ranging just over the four-dimensional spacetime directions α, β . From the ansatz (3.1), the relevant terms in the eleven-dimensional Ricci tensor and Ricci scalar are given by

$$\begin{aligned} \hat{R}_{\alpha\beta} = & \frac{4g^2}{3\tilde{\Delta}^{8/3}} \left[- \left(\sum_i X_i^2 \mu_i^2 \right)^2 + \tilde{\Delta} \sum_i X_i^2 \mu_i^2 \sum_j X_j + \tilde{\Delta} \sum_i X_i^3 \mu_i^2 \right. \\ & \left. - \tilde{\Delta}^2 \sum_i X_i^2 \right] \eta_{\alpha\beta} + \tilde{\Delta}^{-2/3} R_{\alpha\beta} + \dots, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \hat{R} = & \frac{2g^2}{3\tilde{\Delta}^{8/3}} \left[- \left(\sum_i X_i^2 \mu_i^2 \right)^2 - 2\tilde{\Delta} \sum_i X_i^2 \mu_i^2 \sum_j X_j + 4\tilde{\Delta} \sum_i X_i^3 \mu_i^2 \right. \\ & \left. + 6\tilde{\Delta}^2 \left(\sum_i X_i \right)^2 - 7\tilde{\Delta}^2 \sum_i X_i^2 \right] + \tilde{\Delta}^{-2/3} R + \dots, \end{aligned} \quad (3.11)$$

where $R_{\alpha\beta}$ and R are the four-dimensional Ricci tensor and scalar, and the ellipses indicate that terms not involving purely the undifferentiated scalars have been omitted for the purposes of the present illustrative discussion. From the ansatz (3.6) for the 4-form, the eleven-dimensional energy-momentum tensor vielbein components in the four-dimensional spacetime directions are given by

$$\frac{1}{12}(F_{\alpha\beta}^2 - \frac{1}{8}F^2 \eta_{\alpha\beta}) = -g^2 \tilde{\Delta}^{-8/3} \left(\sum_i (X_i^2 \mu_i^2 - \tilde{\Delta} X_i) \right)^2 \eta_{\alpha\beta}. \quad (3.12)$$

Substituting (3.10), (3.11) and (3.12) into (3.9), we find that all the angular dependence coming from the μ_i variables cancels, and that the scalar potential terms in the four-dimensional Einstein equation are given by

$$R_{\alpha\beta} - \frac{1}{2}R \eta_{\alpha\beta} = -\frac{1}{2}g^2 V \eta_{\alpha\beta}, \quad (3.13)$$

with V given by

$$V = -4 \sum_{i < j} X_i X_j = -8(\cosh \varphi_1 + \cosh \varphi_2 + \cosh \varphi_3). \quad (3.14)$$

Since (3.13) derives from the Lagrangian $R - g^2 V$, we see that we have precisely produced the hoped-for potential terms of the gauged supergravity Lagrangian (3.8). This sample calculation also serves to illustrate that the angular dependence coming from the μ_i variables would not have cancelled if we had merely substituted the ansätze (3.1) and (3.6) into the eleven-dimensional Lagrangian. It also shows that the cancellation of the μ_i dependence in the higher-dimensional equations of motion depends crucially on “conspiracies” between the contributions from the metric and the 4-form field strength. This is quite different from the situation in toroidal reductions, where each term in the higher-dimensional theory reduces

consistently by itself, without the need for any such conspiracies. Note, furthermore, that the required conspiracies needed for the success of the spherical reduction depend on the 4-form field strength occurring with precisely the correct coefficient relative to the Einstein-Hilbert term. This normalisation is not a free parameter, but is governed by the strength of the FFA term in the eleven-dimensional theory. Thus ultimately the consistency of the spherical reduction ansatz can be traced back to the supersymmetry of the eleven-dimensional theory.

Note that the Lagrangian (3.8) can be further truncated, to pure Einstein-Maxwell with a cosmological constant, by setting all the field strengths equal, $F_{(2)}^i = \frac{1}{2}F_{(2)}$, and setting all the scalars to zero:

$$e^{-1}\mathcal{L}_4 = R - \frac{1}{4}(F_{(2)})^2 + 24g^2 . \quad (3.15)$$

The embedding of this theory into $D = 11$ supergravity was obtained in [34]. The ansatz for the metric and field strength for the embedding in [34] was given in terms of a decomposition of the 7-sphere as a $U(1)$ bundle over CP^3 . This is identical, after a transformation of coordinates, to the Einstein-Maxwell embedding given in [8]. In the same spirit, the S^5 reduction to five-dimensional Einstein-Maxwell can be described using the method presented in [34], with S^5 viewed as a $U(1)$ bundle over CP^2 . (An analogous consistent embedding of four-dimensional Einstein-Yang-Mills with an $SU(2)$ gauge group, and a cosmological constant, in $D = 11$ supergravity was obtained in [60]. This involves a decomposition of S^7 as an $SU(2)$ bundle over S^4 .)

3.2 $D = 4$ AdS black holes

The $D = 4$, $N = 2$ gauged supergravity coupled to three vector multiplets admits 4-charge AdS black hole solutions, given by [7, 11]

$$\begin{aligned} ds_4^2 &= -(H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3)^{1/2} (f^{-1} dr^2 + r^2 d\Omega_{2,k}^2) , \\ X_i &= H_i^{-1} (H_1 H_2 H_3 H_4)^{1/4} , \quad A_{(1)}^i = \sqrt{k} (1 - H_i^{-1}) \coth \beta_i dt , \end{aligned} \quad (3.16)$$

and

$$f = k - \frac{\mu}{r} + 4g^2 r^2 (H_1 H_2 H_3 H_4) , \quad H_i = 1 + \frac{\mu \sinh^2 \beta_i}{k r} . \quad (3.17)$$

Here, k can be 1, 0 or -1 , corresponding to the cases where the foliations in the transverse space have the metric $d\Omega_{2,k}^2$ on the unit S^2 , T^2 or H^2 , where H^2 is the unit hyperbolic 2-space of constant negative curvature. In the case of $k = 0$, one must first make the rescaling $\sinh^2 \beta_i \rightarrow k \sinh^2 \beta_i$ before sending k to zero. The gauge potential for the $k = 0$ case is

then given by

$$A_{(1)}^i = \frac{1 - H_i^{-1}}{\sinh \beta_i} dt . \quad (3.18)$$

3.3 Rotating M2-brane

There are four angular momenta, ℓ_i , $i = 1, 2, 3, 4$, in the rotating M2-brane. The solution can be obtained by oxidising the $D = 9$ rotating black hole [46]. After the oxidation, we find that the metric of the rotating M2-brane is given by

$$\begin{aligned} ds_{11}^2 = & H^{-\frac{2}{3}} \left(- \left(1 - \frac{2m}{r^6 \Delta} \right) dt^2 + dx_1^2 + dx_2^2 \right) + H^{\frac{1}{3}} \left[\frac{\Delta dr^2}{H_1 H_2 H_3 H_4 - \frac{2m}{r^6}} \right. \\ & + r^2 \sum_{i=1}^4 H_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) - \frac{4m \cosh \alpha}{r^6 H \Delta} dt \left(\sum_{i=1}^4 \ell_i \mu_i^2 d\phi_i \right) \\ & \left. + \frac{2m}{r^4 H \Delta} \left(\sum_{i=1}^4 \ell_i \mu_i^2 d\phi_i \right)^2 \right] , \end{aligned} \quad (3.19)$$

where the functions Δ , H and H_i are given by

$$\begin{aligned} \Delta &= H_1 H_2 H_3 H_4 \sum_{i=1}^4 \frac{\mu_i^2}{H_i} , \quad H = 1 + \frac{2m \sinh^2 \alpha}{r^6 \Delta} , \\ H_i &= 1 + \frac{\ell_i^2}{r^2} , \quad i = 1, 2, 3, 4 . \end{aligned} \quad (3.20)$$

The 3-form gauge potential is given by

$$A_{(3)} = \frac{1 - H^{-1}}{\sinh \alpha} (-\cosh \alpha dt + \ell_i \mu_i^2 d\phi_i) \wedge d^2 x . \quad (3.21)$$

Following the previous D3-brane example, we consider the decoupling limit, which is obtained by making the rescaling

$$\begin{aligned} m &\longrightarrow \epsilon^6 m , \quad \sinh \alpha \longrightarrow \epsilon^{-3} \sinh \alpha , \\ r &\longrightarrow \epsilon r , \quad x^\mu \longrightarrow \epsilon^{-2} x^\mu , \quad \ell_i \longrightarrow \epsilon \ell_i \end{aligned} \quad (3.22)$$

and then sending $\epsilon \longrightarrow 0$. This has the effect that the last term in (3.19) is set to zero and that the 1 in function H (3.20) is removed. In this limit, the rotating M2-brane (3.19) becomes

$$\begin{aligned} ds_{11}^2 = & \tilde{\Delta}^{2/3} \left[- (H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3 H_4)^{1/2} (f^{-1} d\rho^2 + \rho^2 d\vec{y} \cdot d\vec{y}) \right] \\ & + g^{-2} \tilde{\Delta}^{-1/3} \sum_{i=1}^4 X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + g A^i)^2 \right) , \end{aligned} \quad (3.23)$$

where

$$\begin{aligned}\rho &= \tfrac{1}{2} g r^2 , & \vec{y} &= 2g \vec{x} , & f &= -\frac{\mu}{\rho} + 4g^2 \rho^2 H_1 H_2 H_3 H_4 , \\ g^2 &= (2m \sinh^2 \alpha)^{-1/3} , & \mu &= m g^5 , & \tilde{\Delta} &= \sum_i X_i \mu_i^2 .\end{aligned}\quad (3.24)$$

This is precisely of the form of the metric ansatz in the dimensional reduction given by (3.1). The lower dimensional fields are given by

$$\begin{aligned}ds_4^2 &= -(H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3 H_4)^{1/2} (f^{-1} d\rho^2 + \rho^2 d\vec{x} \cdot d\vec{x}) \\ X_i &= H_i^{-1} (H_1 H_2 H_3 H_4)^{1/4} , & A^i &= \frac{1 - H_i^{-1}}{g \ell_i \sinh \alpha} dt .\end{aligned}\quad (3.25)$$

In the decoupling limit, the gauge potential $A_{(3)}$ given in (3.21) for the rotating M2-brane becomes, after a gauge transformation,

$$A_{(3)} = -g^6 r^6 \Delta dt \wedge d^2 x + \frac{1}{\sinh \alpha} \sum_i \ell_i \mu_i^2 d\phi_i \wedge d^2 x . \quad (3.26)$$

We find that its field strength $F_{(4)} = dA_{(3)}$ is also of the form given in (3.6) for the dimensional reduction ansatz. Thus we have established an exact embedding of four-dimensional non-extremal 4-charge AdS black holes into eleven-dimensional supergravity, and furthermore, that they become precisely the decoupling limit of the rotating M2-branes. It should, of course, be emphasised that the four-dimensional AdS black holes that we are considering at this point have T^2 rather than S^2 horizons, corresponding to $k = 0$ in (3.16) and (3.17).

It is also straightforward to oxidise the $k = 1$ and $k = -1$ AdS black hole solutions back to $D = 11$, by substituting the four-dimensional fields into the ansätze (3.1) and (3.6).

4 S^4 reduction of $D = 11$ supergravity

4.1 Reduction ansätze

The Kaluza-Klein reduction of eleven-dimensional supergravity on S^4 gives rise to $N = 4$ gauged $SO(5)$ supergravity in seven dimensions. In a similar manner to the S^5 and S^7 reductions that we discussed previously, we may consider an $N = 2$ truncation of this seven-dimensional theory. As described in the introduction, the truncated theory comprises $N = 2$ supergravity coupled to a vector multiplet, comprising the metric, 2-form potential, four vector potentials and four scalars in total. For our present purposes, we shall focus on a further truncation where only the metric, two gauge potentials (which are associated with the $U(1) \times U(1)$ Cartan subgroup of $SO(5)$) and two scalars are retained. This is not

in general a consistent truncation, but, as in the case of the neglect of the axions in the S^7 reduction, it is consistent for a subset of solutions where the truncated fields are not excited by the ones that are retained. In particular, solutions of the $N = 2$ theory for which $F_{(2)}^1 \wedge F_{(2)}^2 = 0$, such as the AdS black holes, will also be solutions of this truncated theory.

We find that we can obtain this truncated theory by making the following Kaluza-Klein S^4 -reduction ansatz:

$$ds_{11}^2 = \tilde{\Delta}^{1/3} ds_7^2 + g^{-2} \tilde{\Delta}^{-2/3} \left(X_0^{-1} d\mu_0^2 + \sum_{i=1}^2 X_i^{-1} (d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2) \right), \quad (4.1)$$

$$\begin{aligned} *F_{(4)} = & 2g \sum_{\alpha=0}^2 \left(X_\alpha^2 \mu_\alpha^2 - \tilde{\Delta} X_\alpha \right) \epsilon_{(7)} + g \tilde{\Delta} X_0 \epsilon_{(7)} + \frac{1}{2g} \sum_{\alpha=0}^2 X_\alpha^{-1} \bar{*} dX_\alpha \wedge d(\mu_\alpha^2) \\ & + \frac{1}{2g^2} \sum_{i=1}^2 X_i^{-2} d(\mu_i^2) \wedge (d\phi_i + g A_{(1)}^i) \wedge \bar{*} F_{(2)}^i, \end{aligned} \quad (4.2)$$

where we have defined the auxiliary variable $X_0 \equiv (X_1 X_2)^{-2}$. Here, $\bar{*}$ denotes the Hodge dual with respect to the seven-dimensional metric ds_7^2 , $\epsilon_{(7)}$ denotes its volume form, and $*$ denotes the Hodge dualisation in the eleven-dimensional metric. The quantity $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \sum_{\alpha=0}^2 X_\alpha \mu_\alpha^2, \quad (4.3)$$

where μ_0, μ_1 and μ_2 satisfy $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The two scalar fields X_i can be parameterised in terms of two canonically-normalised dilatons $\vec{\varphi} = (\varphi_1, \varphi_2)$ by writing

$$X_i = e^{-\frac{1}{2} \vec{a}_i \cdot \vec{\varphi}}, \quad (4.4)$$

where the dilaton vectors satisfy the relations $\vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - \frac{8}{5}$. A convenient parameterisation is given by

$$\vec{a}_1 = (\sqrt{2}, \sqrt{\frac{2}{5}}), \quad \vec{a}_2 = (-\sqrt{2}, \sqrt{\frac{2}{5}}). \quad (4.5)$$

Note that the two X_i are independent here, unlike in the cases of the three X_i in $D = 5$ or the four X_i in $D = 4$, which satisfied $\prod_i X_i = 1$. The auxiliary variable X_0 that we have introduced in order to make the expressions more symmetrical can be written as $X_0 = e^{-\frac{1}{2} \vec{a}_0 \cdot \vec{\varphi}}$, where $\vec{a}_0 = -2(\vec{a}_1 + \vec{a}_2) = (0, -4\sqrt{2/5})$.

After substituting into the eleven-dimensional equations of motion, one obtains seven-dimensional equations that can be derived from the Lagrangian

$$e^{-1} \mathcal{L}_7 = R - \frac{1}{2} (\partial \vec{\varphi})^2 - g^2 V - \frac{1}{4} \sum_{i=1}^2 e^{\vec{a}_i \cdot \vec{\varphi}} (F_{(2)}^i)^2, \quad (4.6)$$

where the potential V is given by

$$V = -4X_1X_2 - 2X_1^{-1}X_2^{-2} - 2X_2^{-1}X_1^{-2} + \frac{1}{2}(X_1X_2)^{-4} . \quad (4.7)$$

This potential has a more complicated structure than those in the $D = 5$ and $D = 4$ gauged theories, and in particular it has not only a maximum at $X_1 = X_2 = 1$, but also a saddle point at $X_1 = X_2 = 2^{-1/5}$ [61]. Note that by making use of the auxiliary variable $X_0 = (X_1X_2)^{-2}$, the potential can be re-expressed as

$$V = -4X_1X_2 - 2X_0X_1 - 2X_0X_2 + \frac{1}{2}X_0^2 . \quad (4.8)$$

It is interesting to note that the Lagrangian (4.6) can be further consistently truncated, by setting $X_1 = X_2 = X$, and $F_{(2)}^1 = F_{(2)}^2 = F_{(2)}/\sqrt{2}$. This implies that the dilatonic scalar φ_1 is set to zero, in terms of the parameterisation defined by (4.5). This gives

$$e^{-1}\mathcal{L}_7 = R - \frac{1}{2}(\partial\varphi_2)^2 + g^2(4X^2 + 4X^{-3} - \frac{1}{2}X^{-8}) - \frac{1}{4}X^{-2}(F_{(2)})^2 , \quad (4.9)$$

where

$$X = e^{-\frac{1}{\sqrt{10}}\varphi_2} . \quad (4.10)$$

This scalar potential was used in [62] to construct supersymmetric domain wall solutions.

4.2 $D = 7$ AdS black holes

This Lagrangian (4.6) admits 2-charge AdS black-hole solutions, given by

$$\begin{aligned} ds_7^2 &= -(H_1H_2)^{-4/5} f dt^2 + (H_1H_2)^{1/5} (f^{-1} dr^2 + r^2 d\Omega_{5,k}^2) , \\ f &= k - \frac{\mu}{r^4} + \frac{1}{4}g^2 r^2 H_1H_2 , \quad X_i = (H_1H_2)^{2/5} H_i^{-1} , \\ A_{(1)}^i &= \sqrt{k} \coth \beta_i (1 - H_i^{-1}) dt , \quad H_i = 1 + \frac{\mu \sinh^2 \beta_i}{r^4} , \end{aligned} \quad (4.11)$$

where $d\Omega_{5,k}^2$ is the metric on a unit S^5 , T^5 or H^5 according to whether $k = 1, 0$ or -1 . As in the previous cases we discussed, the $k = 0$ solution is obtained by first rescaling $\sinh^2 \beta_i \rightarrow k \sinh^2 \beta_i$ before setting $k = 0$. The metric of the $D = 7$ AdS black hole was obtained in [9], by isolating the spacetime direction of the rotating M5-brane metric.

4.3 Rotating M5-brane

There are two angular momenta, ℓ_1 and ℓ_2 , in the rotating M5-brane [47, 54]. Its metric is given by

$$ds_{11}^2 = H^{-1/3} \left(- \left(1 - \frac{2m}{r^3 \Delta} \right) dt^2 + dx_1^2 + \cdots + dx_5^2 \right) + H^{2/3} \left[\frac{\Delta dr^2}{H_1 H_2 - \frac{2m}{r^3}} \right]$$

$$\begin{aligned}
& +r^2 \left(d\mu_0^2 + \sum_{i=1}^2 H_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) \right) - \frac{4m \cosh \alpha}{r^3 H \Delta} dt \left(\sum_{i=1}^2 \ell_i \mu_i^2 d\phi_i \right)^2 \\
& + \frac{2m}{r^3 H \Delta} \left(\sum_{i=1}^2 \ell_i \mu_i^2 d\phi_i \right)^2 \Big] , \tag{4.12}
\end{aligned}$$

where Δ , H and H_i are given by

$$\begin{aligned}
\Delta &= H_1 H_2 \left(\mu_0^2 + \frac{\mu_1^2}{H_1} + \frac{\mu_2^2}{H_2} \right) , \quad H = 1 + \frac{2m \sinh^2 \alpha}{r^3 \Delta} , \\
H_1 &= 1 + \frac{\ell_1^2}{r^2} , \quad H_2 = 1 + \frac{\ell_2^2}{r^2} . \tag{4.13}
\end{aligned}$$

The three quantities μ_0 , μ_1 and μ_2 satisfy $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The 4-form field strength is given by $F_4 = *dA_6$, where

$$A_6 = \frac{1 - H^{-1}}{\sinh \alpha} (\cosh \alpha dt + \ell_1 \mu_1^2 d\phi_1 + \ell_2 \mu_2^2 d\phi_2) \wedge d^5 x . \tag{4.14}$$

The decoupling limit is defined by

$$\begin{aligned}
m &\longrightarrow \epsilon^3 m , \quad \sinh \alpha \longrightarrow \epsilon^{-3/2} \sinh \alpha , \\
r &\longrightarrow \epsilon r , \quad x^\mu \longrightarrow \epsilon^{-1/2} x^\mu , \quad \ell_i \longrightarrow \epsilon \ell_i , \tag{4.15}
\end{aligned}$$

with $\epsilon \longrightarrow 0$. In this limit, the metric becomes

$$\begin{aligned}
ds_{11}^2 &= \tilde{\Delta}^{1/3} \left[- (H_1 H_2)^{-4/5} f dt^2 + (H_1 H_2)^{1/5} (f^{-1} d\rho^2 + \rho^2 d\vec{y} \cdot d\vec{y}) \right] \\
&+ g^{-2} \tilde{\Delta}^{-2/3} \left((X_1 X_2)^2 d\mu_0^2 + \sum_{i=1}^2 X_i^{-1} (d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2) \right) . \tag{4.16}
\end{aligned}$$

where

$$\begin{aligned}
\rho^2 &= 4r g^{-1} , \quad \vec{y} = \tfrac{1}{2} g \vec{x} , \quad \tilde{\Delta} = (X_1 X_2)^{-2} \mu_0^2 + X_1 \mu_1^2 + X_2 \mu_2^2 , \\
g^2 &= (2m \sinh^2 \alpha)^{-2/3} , \quad \mu = 32m g^{-1} . \tag{4.17}
\end{aligned}$$

The metric (4.16) fits precisely the dimensional reduction ansatz given in (4.1). The lower dimensional fields are given by

$$\begin{aligned}
ds_7^2 &= - (H_1 H_2)^{-4/5} f dt^2 + (H_1 H_2)^{1/5} (f^{-1} d\rho^2 + \rho^2 d\vec{y} \cdot d\vec{y}) , \\
X_i &= (H_1 H_2)^{2/5} H_i^{-1} , \quad f = -\frac{\mu}{\rho^4} + \tfrac{1}{4} g^2 \rho^2 H_1 H_2 , \\
A_{(1)}^i &= \frac{1 - H_i^{-1}}{g \ell_i \sinh \alpha} dt . \tag{4.18}
\end{aligned}$$

This is precisely the $k = 0$ AdS₇ black hole obtained in previous section, with the angular momenta reparameterised as $\ell_i = \mu g^2 \sinh^2 \beta_i / 16$. This establishes that the 2-charge $k = 0$

AdS black hole in $D = 7$ can be reinterpreted as the decoupling limit of the rotating M2-brane. (Of course in this example, one can *only* discuss the embedding when the scalar fields are included, since there is no choice of charge parameters for which the scalar fields vanish in the seven-dimensional black holes. This contrasts with the cases of the rotating D3-branes and M5-branes, where the special choice of setting all the charges equal allows the discussion of a simplified ansatz where the scalars are omitted.)

5 Sphere reduction of generic rotating p -branes, and domain wall black holes

The general expression for a rotating p -brane carrying a single charge is given in appendix A. Following the procedure in the previous sections, we may take the limit of large p -brane charge, by performing the rescalings

$$\begin{aligned} m &\longrightarrow \epsilon^{\tilde{d}} m, & \sinh \alpha &\longrightarrow \epsilon^{-\frac{\tilde{d}}{2}} \sinh \alpha, \\ r &\longrightarrow \epsilon r, & x^\mu &\longrightarrow \epsilon^{1-\tilde{d}/2} x^\mu, & \ell_i &\longrightarrow \epsilon \ell_i, \end{aligned} \quad (5.1)$$

and then sending ϵ to zero. We find that the metric becomes $ds^2 = \epsilon^{a^2/2} d\tilde{s}^2$, where a is given under (A.1) and the metric $d\tilde{s}^2$ is given by

$$\begin{aligned} d\tilde{s}_D^2 &= \tilde{\Delta}^{\frac{\tilde{d}}{D-2}} e^{\tilde{d}\varphi} \left[- (H_1 \cdots H_N)^{-\frac{d-2}{d-1}} f dt^2 \right. \\ &\quad \left. + (H_1 \cdots H_N)^{\frac{1}{d-1}} \left(\left(\frac{\tilde{d}}{d} g \rho \right)^{\frac{(D-2)a^2}{2d}} f^{-1} d\rho^2 + \rho^2 d\vec{y} \cdot d\vec{y} \right) \right] \\ &\quad + g^{-2} \tilde{\Delta}^{-\frac{d}{D-2}} e^{-d\varphi} \sum_{i=1}^N X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + g A^i)^2 \right), \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} g \rho &= (d/\tilde{d}) (g r)^{\tilde{d}/d}, & \vec{y} &= g (\tilde{d}/d) \vec{x}, \\ g^{-\tilde{d}} &= 2m \sinh^2 \alpha, & \mu &= 2m (d/\tilde{d})^{d-2} g^{2+\tilde{d}-d}, \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} f &= -\frac{\mu}{\rho^{d-2}} + (\tilde{d}/d)^2 g^2 \rho^2 (H_1 \cdots H_N), & X_i &= (H_1 \cdots H_N)^{\frac{d}{(d-1)\tilde{d}}} H_i^{-1}, \\ \tilde{\Delta} &= \frac{\sum_i X_i \mu_i^2}{(X_1 \cdots X_N)^2}, & e^{-\frac{2\tilde{d}}{a^2}\varphi} &= (\tilde{d}/d) g \rho, & A^i &= \frac{1 - H_i^{-1}}{g \ell_i \sinh \alpha} dt. \end{aligned} \quad (5.4)$$

It follows that the $(d+1)$ dimensional metric becomes

$$ds_{d+1}^2 = - (H_1 \cdots H_N)^{-\frac{d-2}{d-1}} f dt^2 + (H_1 \cdots H_N)^{\frac{1}{d-1}} \left(e^{-(D-2)\varphi} f^{-1} d\rho^2 + \rho^2 d\vec{y} \cdot d\vec{y} \right). \quad (5.5)$$

The Einstein-frame metric is given by $ds_E^2 = e^{-\frac{(D-2)}{(d-1)}\varphi} ds_{d+1}^2$. This is the metric of an N -charge black hole in a domain-wall background. In the case when $a = 0$, the domain wall specialises to AdS_{d+1} .

6 Conclusions

In this paper, we have constructed the non-linear ansätze for the spherical dimensional reduction of type IIB supergravity on S^5 , and eleven-dimensional supergravity on S^7 and S^4 , in the case where we restrict to the abelian subgroups of the full $SO(6)$, $SO(8)$ and $SO(5)$ gauge groups. In this way, we have shown how the gauged theories in $D = 5$, $D = 4$ and $D = 7$ that are relevant for constructing charged AdS black holes can be embedded into type IIB supergravity or eleven-dimensional supergravity.

As a matter of fact, in order to work out what the non-linear metric and field-strength ansätze should be, we got many hints by looking at the detailed forms of the lower-dimensional configurations that one obtains by making the appropriate sphere compactifications of the near-horizon limits of the corresponding spinning D3-brane, M2-brane or M5-brane, and comparing the results with the AdS black-hole solutions in $D = 5$, $D = 4$ and $D = 7$. The key step in being able to extract general results for the reduction ansätze from these specific solutions is that one must first establish that the various components of the higher-dimensional metrics and field strengths can in fact be expressed in a generic way in terms of the fields of the lower-dimensional theory, together with the coordinates of the compactifying sphere. Only by doing this can one then “kick away the ladder,” and extract general results, independent of any *specific* solution, for how an *arbitrary* solution of the lower-dimensional equations of motion can be embedded in the higher-dimensional theory. Luckily, the multi-charge AdS black-hole solutions are general enough that they provided many clues that were helpful in deducing what the full ansätze should be.

Rather general, although highly implicit, results had previously been given for the metric ansatz for the full $SO(8)$ reduction of $D = 11$ supergravity on S^7 [35, 36]. In principle, it should be possible to verify that the $U(1)^4$ truncation of the general case is in agreement with our result (3.1) for the $U(1)^4$ gauged theory. In practice, however, the implicit nature of the general expressions in [35, 36] makes the comparison rather difficult. The situation regarding the 4-form field strength is even less clear, and only results of an even more implicit nature have been previously presented. Even less has been given previously for the other examples, namely the S^5 and S^4 reductions. It should be emphasised that it is the

handling of the scalar fields in the various ansätze that presents the major challenges in performing spherical Kaluza-Klein reductions.

As far as we are aware, therefore, the results in this paper provide the first explicit examples of compactifications of $D = 11$ supergravity on S^4 and S^7 , and type IIB supergravity on S^5 , in which sets of lower-dimensional massless fields that include scalars are embedded in the reduction ansätze. Importantly, we showed that the equivalence between the corresponding gauged supergravities and the compactified higher-dimensional theory is at the level of the equations of motion, rather than at the level of the effective Lagrangian. Furthermore, the fact that one is able at all to read off sensible lower-dimensional equations of motion depends crucially on conspiracies between the contributions of the ansätze for the higher-dimensional metric and antisymmetric tensor. (Without such conspiracies, one would not get a clean factorisation of lower-dimensional equations of motion multiplied by overall sphere-dependent factors.) This emphasises that non-trivial spherical reductions, in which Kaluza-Klein gauge fields and scalars are retained, make sense only in the context of certain very special higher-dimensional theories. All the known examples of such theories are supergravities.

Having found the non-linear Kaluza-Klein ansätze, we were able to provide an explicit demonstration of how the multi-charge AdS black holes in gauged $D = 5$, $D = 4$ and $D = 7$ supergravities that have toroidal horizons are embedded in type IIB supergravity or $D = 11$ supergravity, and to show that the higher-dimensional solutions are precisely the near-horizon decoupling limits of spinning D3-branes, M2-branes and M5-branes. (Previous partial results for a single-charge AdS_5 black hole appeared in [9], and results for the special case of Reissner-Nordström AdS_5 and AdS_4 black holes appeared in [8].)

The results that we have obtained have also opened the door to the study of the embedding into M-theory or string theory of other solutions of gauged supergravities in $D = 4$, $D = 5$ and $D = 7$ in dimensions; for example AdS black holes with other topologies (such as spheres or hyperbolic spaces), strings, domain walls, *etc.* These solutions could in turn provide novel information about other possible distortions of the spherical compactifications (not only those related to rotations), and thus provide new insights into strongly-coupled gauged theories via the AdS/CFT correspondence.

Acknowledgements

We made extensive use of the TTC tensor-manipulation package. J.T.L. and C.N.P. thank the University of Pennsylvania for hospitality, and J.T.L. thanks Texas A&M University for hospitality.

Appendices

A Single-charge rotating p -branes

In this appendix, we present, for convenience, some general results for rotating p -branes in arbitrary dimensions, supported by a single $(p+2)$ -form charge. These are all straightforwardly obtained by diagonally oxidising the rotating black holes constructed in [46].

Single-charge p -branes in supergravity theories are solutions of the Lagrangian

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!} e^{a\phi} (F_{(n)})^2, \quad (\text{A.1})$$

where $F_{(n)} = dA_{(n-1)}$ and $a^2 = 4 - 2(n-1)(D-n-1)/(D-2)$ [63]. In this appendix, we obtain rotating p -brane solutions. The Lagrangian (A.1) admits an electric $(d-1)$ -brane with $d = n-1$ or a magnetic $(d-1)$ -brane with $d = D-n-1$. We shall consider only the electric solution here, since the magnetic one can be viewed as an electric solution of the dual $(D-n)$ -form field strength $F_{(D-n)}$. The rotating p -brane can be dimensionally reduced on its world-volume spatial coordinates, to give rise to single-charge rotating black holes, which were obtained in [46]. Conversely, it is a straightforward procedure dimensionally to oxidise the rotating black hole solutions in [46] to give the rotating p -branes in higher dimensions. We shall use this approach to obtain general single-charge rotating p -branes in this appendix.

Introducing a dual parameter $\tilde{d} = D-d-2 = D-n-1$, the dimension of the transverse space is $\tilde{d}+2$. It follows that the foliating spheres of the transverse space have dimension $\tilde{d}+1$. There are two cases arising, depending on whether \tilde{d} is even or odd.

Case 1: $\tilde{d}+2 = 2N$

In this case, there are N angular momenta ℓ_i , with $i = 1, 2, \dots, N$. We find that the metric of the rotating $(n-2)$ -brane solution to the equations following from (A.1) is

$$ds_D^2 = H^{-\frac{\tilde{d}}{D-2}} \left(- \left(1 - \frac{2m}{r^{\tilde{d}}\Delta} \right) dt^2 + d\vec{x} \cdot d\vec{x} \right) + H^{\frac{d}{D-2}} \left[\frac{\Delta dr^2}{H_1 \cdots H_N - 2m r^{-\tilde{d}}} \right]$$

$$\begin{aligned}
& + r^2 \sum_{i=1}^N H_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) - \frac{4m \cosh \alpha}{r^{\tilde{d}} H \Delta} dt \left(\sum_{i=1}^N \ell_i \mu_i^2 d\phi_i \right) \\
& + \frac{2m}{r^{\tilde{d}} H \Delta} \left(\sum_{i=1}^N \ell_i \mu_i^2 d\phi_i \right)^2 \Big] , \tag{A.2}
\end{aligned}$$

where the functions Δ , H and H_i are given by

$$\begin{aligned}
\Delta &= H_1 \cdots H_N \sum_{i=1}^N \frac{\mu_i^2}{H_i} , \quad H = 1 + \frac{2m \sinh^2 \alpha}{r^{\tilde{d}} \Delta} , \\
H_i &= 1 + \frac{\ell_i^2}{r^2} , \quad i = 1, 2, \dots, N . \tag{A.3}
\end{aligned}$$

The dilaton ϕ and gauge potential $A_{(n-1)}$ are given by

$$e^{2\phi/a} = H , \quad A_{(n-1)} = \frac{1 - H^{-1}}{\sinh \alpha} \left(\cosh \alpha dt + \sum_{i=1}^N \ell_i \mu_i^2 d\phi_i \right) \wedge d^{n-2}x . \tag{A.4}$$

The N quantities μ_i , as usual, are subject to the constraint $\sum_i \mu_i^2 = 1$. One can parameterise the μ_i in terms of $(N-1)$ unconstrained angles. A common choice is

$$\begin{aligned}
\mu_i &= \sin \psi_i \prod_{j=1}^{i-1} \cos \psi_j , \quad i \leq N-1 , \\
\mu_N &= \prod_{j=1}^{N-1} \cos \psi_j . \tag{A.5}
\end{aligned}$$

Note that $\prod_{j=1}^n \cos \psi_j$ is defined to be equal to 1 if $n \leq 0$.

Case 2: $\tilde{d} + 2 = 2N + 1$

Here, the solution has the same form as in Case 1, but with the range of the index i extended to include 0. (Note that our variable μ_0 is called α in [57].) However, there is no angular momentum parameter or azimuthal coordinate associated with the extra index value, and so $\ell_0 = 0$ and $\phi_0 = 0$. Otherwise, all the formulae in Case 1 are generalised simply by extending the summation to span the range $0 \leq i \leq N$. Of course $H_0 = 1$ as a consequence of $\ell_0 = 0$.

B $D = 4$, $N = 2$ gauged supergravity

The $SO(8)$ gauged $N = 8$ supergravity in four dimensions was obtained in [17, 18] by gauging an $SO(8)$ subgroup of the global E_7 symmetry group of [64, 65]. To avoid some of the complications of non-abelian gauge fields, one may consider a truncation of this

model to $N = 2$, for which the bosonic sector comprises the metric, four commuting $U(1)$ gauge potentials, three dilatons and three axions. In the absence of axions, this truncation was obtained in [7] by working in the symmetric gauge for the 56-bein and incorporating three real scalars. As was noted, there is a straightforward generalization of the scalar ansatz to allow for complex scalars. Taking into account the E_7 self-duality condition $\overline{\phi}^{ijkl} = \phi_{ijkl} = \frac{1}{4!} \epsilon_{ijklmnpq} \phi^{ijkl}$, the scalar ansatz of [7] may be generalized as:

$$\overline{\phi}^{ijkl} = \phi_{ijkl} = \sqrt{2} [\Phi^{(1)} \epsilon^{(12)} + \Phi^{(2)} \epsilon^{(13)} + \Phi^{(3)} \epsilon^{(14)} + \overline{\Phi}^{(1)} \epsilon^{(34)} + \overline{\Phi}^{(2)} \epsilon^{(24)} + \overline{\Phi}^{(3)} \epsilon^{(23)}]_{ijkl}, \quad (\text{B.1})$$

where we follow the notation and conventions of [7] (including the definition of $SO(8)$ index pairs). Note that the three complex scalars may be parameterised in terms of their magnitudes and phases as $\Phi^{(i)} = \phi^{(i)} e^{i\theta^{(i)}}$.

Here, we shall consider the full $N = 2$ truncation, where the three axions are included as well as the other fields. In fact the structure of the potential is little changed. We find that the Lagrangian including the axions may be written in the form

$$e^{-1} \mathcal{L}_4 = R - \frac{1}{2} \sum_i \left((\partial \phi^{(i)})^2 + \sinh^2 \phi^{(i)} (\partial \theta^{(i)})^2 \right) - \frac{1}{2} (F_{\mu\nu}^{(\alpha)+} \mathcal{M}_{\alpha\beta} F^{(\beta)+\mu\nu} + \text{h.c.}) - g^2 V, \quad (\text{B.2})$$

where the potential is given simply by

$$V = -8(\cosh \phi^{(1)} + \cosh \phi^{(2)} + \cosh \phi^{(3)}). \quad (\text{B.3})$$

The complex symmetric scalar matrix \mathcal{M} is quite complicated, and incorporates all three complex scalars $\Phi^{(\alpha)}$ in a symmetric manner; this is presented below.

In terms of the $N = 2$ truncation, the three complex scalars each parameterise an $SL(2; R)/SO(2)$ coset. This may be made explicit by performing the change of variables $(\phi^{(i)}, \theta^{(i)}) \rightarrow (\varphi_i, \chi_i)$:

$$\begin{aligned} \cosh \phi^{(i)} &= \cosh \varphi_i + \frac{1}{2} \chi_i^2 e^{\varphi_i}, \\ \cos \theta^{(i)} \sinh \phi^{(i)} &= \sinh \varphi_i - \frac{1}{2} \chi_i^2 e^{\varphi_i}, \\ \sin \theta^{(i)} \sinh \phi^{(i)} &= \chi_i e^{\varphi_i}. \end{aligned} \quad (\text{B.4})$$

Defining the dilaton-axion combinations

$$A_i = 1 + \chi_i^2 e^{2\varphi_i}, \quad (\text{B.5})$$

as well as

$$\begin{aligned} B_1 &= \chi_2 \chi_3 e^{\varphi_2 + \varphi_3} + i \chi_1 e^{\varphi_1}, \\ B_2 &= \chi_1 \chi_3 e^{\varphi_1 + \varphi_3} + i \chi_2 e^{\varphi_2}, \\ B_3 &= \chi_1 \chi_2 e^{\varphi_1 + \varphi_2} + i \chi_3 e^{\varphi_3}, \end{aligned} \quad (\text{B.6})$$

we finally obtain the bosonic Lagrangian

$$e^{-1}\mathcal{L}_4 = R - \frac{1}{2} \sum_i \left((\partial\varphi_i)^2 + e^{2\varphi_i} (\partial\chi_i)^2 \right) - \frac{1}{2} (F_{\mu\nu}^{(\alpha)+} \mathcal{M}_{\alpha\beta} F^{(\beta)+\mu\nu} + \text{h.c.}) - g^2 V . \quad (\text{B.7})$$

The potential V is now given by

$$V = -8 \sum_i \left(\cosh \varphi_i + \frac{1}{2} \chi_i^2 e^{\varphi_i} \right) , \quad (\text{B.8})$$

and the scalar matrix is

$$\mathcal{M} = \frac{1}{D} \begin{bmatrix} e^{-\lambda_1} & e^{\varphi_1} B_1 & e^{\varphi_2} B_2 & e^{\varphi_3} B_3 \\ e^{\varphi_1} B_1 & e^{-\lambda_2} A_2 A_3 & -e^{-\varphi_3} A_3 B_3 & -e^{-\varphi_2} A_2 B_2 \\ e^{\varphi_2} B_2 & -e^{-\varphi_3} A_3 B_3 & e^{-\lambda_3} A_1 A_3 & -e^{-\varphi_1} A_1 B_1 \\ e^{\varphi_3} B_3 & -e^{-\varphi_2} A_2 B_2 & -e^{-\varphi_1} A_1 B_1 & e^{-\lambda_4} A_1 A_2 \end{bmatrix} , \quad (\text{B.9})$$

where

$$D = 1 + \chi_1^2 e^{2\varphi_1} + \chi_2^2 e^{2\varphi_2} + \chi_3^2 e^{2\varphi_3} - 2i \chi_1 \chi_2 \chi_3 e^{\varphi_1 + \varphi_2 + \varphi_3} . \quad (\text{B.10})$$

The scalar combinations $\{\lambda\}$ are defined as in [7]:

$$\begin{aligned} \lambda_1 &= -\varphi_1 - \varphi_2 - \varphi_3 , \\ \lambda_2 &= -\varphi_1 + \varphi_2 + \varphi_3 , \\ \lambda_3 &= \varphi_1 - \varphi_2 + \varphi_3 , \\ \lambda_4 &= \varphi_1 + \varphi_2 - \varphi_3 . \end{aligned} \quad (\text{B.11})$$

While this $N = 2$ truncation of the $N = 8$ theory essentially treats all four $U(1)$ gauge fields equally, it was noted that one can make contact with the theory obtained by reduction of a closed string on T^2 through dualisation of two of the gauge fields. To be specific, we dualise $F_{\mu\nu}^{(2)}$ and $F_{\mu\nu}^{(4)}$, which singles out the dilaton-axion pair $S = \chi_2 + ie^{-\varphi_2}$. After an additional field redefinition $S \rightarrow -1/S$, we obtain the bosonic Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_4^{\text{dualized}} &= R - \frac{1}{2}(\partial\varphi_2)^2 - \frac{1}{2}e^{2\varphi_2}(\partial\chi_2)^2 + \frac{1}{8}\text{Tr}(\partial M L \partial M L) - g^2 V \\ &\quad - \frac{1}{4}e^{-\varphi_2} F^T (L M L) F - \frac{1}{4}\chi_2 F^T L^* F , \end{aligned} \quad (\text{B.12})$$

where the potential is still given by (B.8). The scalar matrix M is given in terms of the $SL(2; R) \times SL(2; R)$ vielbein

$$\mathcal{V} = e^{\varphi_3/2} \begin{bmatrix} 1 & -\chi_3 \\ 0 & e^{-\varphi_3} \end{bmatrix} \otimes e^{\varphi_1/2} \begin{bmatrix} 1 & -\chi_1 \\ 0 & e^{-\varphi_1} \end{bmatrix} , \quad (\text{B.13})$$

by $M = \mathcal{V}^T \mathcal{V}$, and the gauge fields have been arranged in the particular order

$$F_{\mu\nu} = [F_{\mu\nu}^{(3)} \quad \tilde{F}_{\mu\nu}^{(4)} \quad \tilde{F}_{\mu\nu}^{(2)} \quad -F_{\mu\nu}^{(1)}]^T . \quad (\text{B.14})$$

Finally, $L = \sigma^2 \otimes \sigma^2$ satisfies $L^2 = I_4$ where σ^2 is the standard Pauli matrix. It is worth mentioning that the pure scalar Lagrangian can be expressed as

$$e^{-1} \mathcal{L}_{\text{scalar}} = \sum_{i=1}^3 \left[-\frac{1}{2} \text{tr} \partial \mathcal{M}_i \partial \mathcal{M}_i^{-1} + 4g^2 \text{tr} \mathcal{M}_i \right], \quad (\text{B.15})$$

where $\mathcal{M}_i = \mathcal{V}_i^T \mathcal{V}_i$ and \mathcal{V}_i is given by

$$\mathcal{V}_i = e^{\varphi_i/2} \begin{bmatrix} 1 & -\chi_i \\ 0 & e^{-\varphi_i} \end{bmatrix}. \quad (\text{B.16})$$

We see that, save for the potential, the dualise Lagrangian is indeed of the form obtained from T^2 compactification from six dimensions. In this case, two of the $SL(2; R)$'s now correspond to T -dualities while the third corresponds to S -duality. Note that the initial choice of which two field strengths to dualise has determined which of the three dilaton-axion pairs (φ_i, χ_i) is to be identified with the strong-weak coupling $SL(2; R)$.

Having shown that the bosonic Lagrangian is considerably simplified by dualising to the field variables that arise in the T^2 reduction, we may re-express the result (B.12) in the more explicit notation of [39, 66]. Thus the bosonic sector of the gauged $U(1)^4$ theory may be written as

$$\begin{aligned} e^{-1} \mathcal{L}_4 = & R - \frac{1}{2} (\partial \vec{\varphi})^2 - \frac{1}{2} e^{-\vec{a} \cdot \vec{\varphi}} (\partial \chi)^2 - \frac{1}{2} e^{\vec{a}_{12} \cdot \vec{\varphi}} (F_{(1)12})^2 - \frac{1}{2} e^{\vec{b}_{12} \cdot \vec{\varphi}} (\mathcal{F}_{(1)2}^1)^2 - g^2 V \\ & - \frac{1}{4} \sum_{i=1}^2 \left(e^{\vec{a}_i \cdot \vec{\varphi}} (F_{(2)i})^2 + e^{\vec{b}_i \cdot \vec{\varphi}} (\mathcal{F}_{(2)}^i)^2 \right) - \frac{1}{2} \chi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu i} \mathcal{F}_{\rho\sigma}^i, \end{aligned} \quad (\text{B.17})$$

where the field strengths are given by

$$\begin{aligned} F_{(2)1} &= dA_{(1)1} + dA_{(0)12} \mathcal{A}_{(1)}^2, & \mathcal{F}_{(2)}^1 &= d\mathcal{A}_{(2)}^1 - d\mathcal{A}_{(0)2}^1 \mathcal{A}_{(1)}^2, \\ F_{(2)2} &= dA_{(1)2} - \mathcal{A}_{(0)2}^1 dA_{(1)1} - dA_{(0)12} \mathcal{A}_{(1)}^1, & \mathcal{F}_{(2)}^2 &= d\mathcal{A}_{(1)}^2. \end{aligned} \quad (\text{B.18})$$

Here χ , $A_{(0)12}$ and $\mathcal{A}_{(0)2}^1$ are the axions χ_2 , χ_1 and χ_3 , and the potential is given by (B.8).

The inclusion of the potential term in the gauged supergravity theory breaks all three $SL(2; R)$ symmetries to $O(2)$, acting as $\tau \rightarrow (a\tau + b)/(c\tau + d)$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \alpha/2 & \sin \alpha/2 \\ -\sin \alpha/2 & \cos \alpha/2 \end{pmatrix}. \quad (\text{B.19})$$

In terms of the original (ϕ, θ) scalar variables in (B.2), this $O(2)$ subgroup corresponds to $\theta \rightarrow \theta + \alpha$. The $O(2)$ symmetry is, however, sufficient for generating dyonic solutions. Nevertheless, we note that the fermionic sector and in particular the supersymmetry transformations are not invariant under this symmetry of the bosonic sector. One manifestation

of this particular situation is the fact that magnetic black holes of this theory are not supersymmetric, even though they may be extremal. Furthermore, in a related note, while it is clear that the dualisation procedure performed above runs into difficulty in the full $N = 8$ theory with non-abelian $SO(8)$ gauging, the fermionic sector does not admit such a straightforward dualisation, even in the $N = 2$ abelian truncation. This is easily seen by the fact that the gravitini are necessarily charged under the gauge fields and hence couple to the bare gauge potentials themselves.

C Calculation of the Ricci tensor for the reduction ansätze

Many of the calculations involved in the spherical reduction ansätze in this paper are quite involved, and some of them are more conveniently performed by computer. However, some of them are quite tractable by hand calculation. Here, we present some useful results for some of the curvature calculations for the metric ansatz, which can be presented in a relatively compact form if further specialisations are made, as discussed below.

The general Kaluza-Klein ansatz for odd sphere S^{2k-1} reductions of the D -dimensional metric may be expressed in the form

$$ds_D^2 = \tilde{\Delta}^a ds_d^2 + \tilde{\Delta}^{-b} \sum_{i=1}^k X_i^{-1} \left(d\mu_i^2 + \mu_i^2 (d\phi_i + A_{(1)}^i)^2 \right), \quad (\text{C.1})$$

where we have set the radius of S^{2k-1} to unity. There are $k - 1$ scalar degrees of freedom parameterised by the k quantities X_i satisfying the constraint $\prod_{i=1}^k X_i = 1$. This form of the line element encompasses both the S^5 reduction of type IIB supergravity and the S^7 reduction of eleven dimensional supergravity. As defined previously, $\tilde{\Delta} = \sum_{i=1}^k X_i \mu_i^2$ and $\sum_{i=1}^k \mu_i^2 = 1$.

In the absence of the gauge fields, this metric has a block diagonal form, with the blocks corresponding to the d -dimensional spacetime, the $k - 1$ direction cosines μ_i and the k azimuthal rotation angles ϕ_i . The main difficulty in computing the curvature of (C.1) lies in the fact that the μ_i 's are constrained. Nevertheless, we may perform an asymmetric choice of using the first $k - 1$ of them as actual coordinates, while expressing μ_k as the constrained quantity $\mu_k = (1 - \sum_{i=1}^{k-1} \mu_i^2)^{1/2}$.

Since numerous terms are involved in the computation, it is imperative to clarify our notation. We denote the lower-dimensional spacetime indices by $\mu, \nu, \dots = 0, 1, \dots, d - 1$, the direction cosine indices by $\alpha, \beta, \gamma, \dots = 1, 2, \dots, k - 1$ and the azimuthal indices by $i, j, \dots = 1, 2, \dots, k$. Note that for instance implicit sums over α always run over $k - 1$

values, while sums over i always run over the full k values.

Thus (with vanishing gauge fields) the D -dimensional metric may be expressed in the form

$$G_{MN} = \text{diag}[\tilde{\Delta}^a g_{\mu\nu}, \tilde{\Delta}^{-b} \hat{g}_{ij}, \tilde{\Delta}^{-b} \tilde{g}_{\alpha\beta}] , \quad (\text{C.2})$$

where $\hat{g}_{ij} = X_i^{-1} \mu_i^2 \delta_{ij}$ is diagonal and

$$\begin{aligned} \tilde{g}_{\alpha\beta} &= X_\alpha^{-1} \delta_{\alpha\beta} + X_k^{-1} \hat{\mu}_\alpha \hat{\mu}_\beta , \\ \tilde{g}^{\alpha\beta} &= X_\alpha \delta_{\alpha\beta} - \tilde{\Delta}^{-1} X_\alpha X_\beta \mu_\alpha \mu_\beta , \end{aligned} \quad (\text{C.3})$$

with $\hat{\mu}_\alpha \equiv \mu_\alpha / \mu_k$. Note that $\det \tilde{g}_{\alpha\beta} = \tilde{\Delta} / \mu_k^2$. As the μ_α themselves are coordinates, this allows the use of expressions such as $\partial_\alpha \mu_k = -\hat{\mu}_\alpha$ and $\partial_\alpha \hat{\mu}_\beta = \mu_k^{-1} (\delta_{\alpha\beta} + \hat{\mu}_\alpha \hat{\mu}_\beta)$. In addition, all α, β, \dots indices are raised and lowered with the metric $\tilde{g}_{\alpha\beta}$.

Using this specific form of the metric $\tilde{g}_{\alpha\beta}$ and the fact that $\det \hat{g}_{ij} = \prod_{i=1}^k \mu_i^2$, we find $\det G_{MN} = \det g_{\mu\nu} \tilde{\Delta}^{\kappa+2a} \prod_{i=1}^{k-1} \mu_i^2$ where the product provides the measure over the internal S^{2k-1} . Here $\kappa = a(d-2) - b(2k-1) + 1$ so that $\sqrt{-G}R \sim \sqrt{-g} \tilde{\Delta}^{\kappa/2}$. Hence one expects $\kappa = 0$ in order to prevent any $\tilde{\Delta}$ dependence from appearing in front of the lower-dimensional Einstein term. Indeed we see that κ vanishes for both the S^5 and the S^7 reductions considered in the text.

We have only computed selected components of the full D -dimensional Ricci tensor which are of interest in the Kaluza-Klein reduction. While we have used an asymmetrical parameterisation of the direction cosines, the final results are symmetrical in all k of the μ_i 's. For the lower-dimensional components of R_{MN} we find

$$\begin{aligned} R_{\mu\nu} &= R_{\mu\nu}^{(d)} - \frac{1}{2} \kappa \tilde{\Delta}^{-1} \partial_\mu \partial_\nu \tilde{\Delta} + \frac{1}{4} ((a+2)\kappa - (a+b)b(2k-1) + a+2b) \tilde{\Delta}^{-2} \partial_\mu \tilde{\Delta} \partial_\nu \tilde{\Delta} \\ &\quad + \frac{1}{4} (\partial_\mu \tilde{g}^{\alpha\beta} \partial_\nu \tilde{g}_{\alpha\beta} - X_i^{-2} \partial_\mu X_i \partial_\nu X_i) \\ &\quad + g_{\mu\nu} [-\frac{a}{2} \tilde{\Delta}^{-1} \nabla^2 \tilde{\Delta} - \frac{a}{4} (\kappa-2) \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta}] \\ &\quad + g_{\mu\nu} \tilde{\Delta}^{a+b} [-\frac{a}{4} (\kappa+2a+2b-3) \tilde{\Delta}^{-2} \partial^\alpha \tilde{\Delta} \partial_\alpha \tilde{\Delta} - \frac{a}{2} (\tilde{\Delta}^{-1} \tilde{\nabla}^2 \tilde{\Delta} + \tilde{\Delta}^{-1} \mu_i^{-1} \partial^\alpha \mu_i \partial_\alpha \tilde{\Delta})] , \end{aligned} \quad (\text{C.4})$$

where $R_{\mu\nu}^{(d)}$ denotes the Ricci tensor of the d -dimensional spacetime metric $g_{\mu\nu}$. We have also determined the internal Ricci components R_{ij} and $R_{\alpha\beta}$ necessary for computing the D -dimensional scalar curvature. For the former we find

$$\begin{aligned} R_{ij} &= \hat{g}_{ij} \tilde{\Delta}^{-a-b} [\frac{1}{4} \kappa \tilde{\Delta}^{-1} X_i^{-1} \partial^\rho \tilde{\Delta} \partial_\rho X_i + \frac{b}{2} \tilde{\Delta}^{-1} \nabla^2 \tilde{\Delta} + \frac{b}{4} (\kappa-2) \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta}] \\ &\quad + \hat{g}_{ij} [-\frac{1}{2} (\kappa+2a+2b-1) \tilde{\Delta}^{-1} \mu_i^{-1} \partial^\alpha \mu_i \partial_\alpha \tilde{\Delta} - \mu_i^{-1} \tilde{\nabla}^2 \mu_i + \tilde{\Delta}^{-1} (X_i \sum X - X_i^2) \\ &\quad + \frac{b}{4} (\kappa+2a+2b-3) \tilde{\Delta}^{-2} \partial^\alpha \tilde{\Delta} \partial_\alpha \tilde{\Delta} + \frac{b}{2} \tilde{\Delta}^{-1} \tilde{\nabla}^2 \tilde{\Delta} + \frac{b}{2} \tilde{\Delta}^{-1} \mu_l^{-1} \partial^\alpha \mu_l \partial_\alpha \tilde{\Delta}] \end{aligned} \quad (\text{C.5})$$

(no sum on i), while for the latter we have

$$\begin{aligned}
R_{\alpha\beta} = & \tilde{\Delta}^{-a-b} [\frac{1}{2} \tilde{g}^{\gamma\delta} \partial^\rho \tilde{g}_{\alpha\gamma} \partial_\rho \tilde{g}_{\beta\delta} - \frac{1}{4} \kappa \tilde{\Delta}^{-1} \partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{\Delta} - \frac{1}{2} \nabla^2 \tilde{g}_{\alpha\beta}] \\
& + \tilde{g}_{\alpha\beta} \tilde{\Delta}^{-a-b} [\frac{b}{4} (\kappa - 2) \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta} + \frac{b}{2} \tilde{\Delta}^{-1} \nabla^2 \tilde{\Delta}] \\
& + \tilde{R}_{\alpha\beta} - \frac{1}{2} (\kappa + 2a + 2b - 1) \tilde{\Delta}^{-1} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\Delta} - \mu_i^{-1} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \mu_i \\
& - \frac{1}{4} ((b-2)\kappa + a(a+b)d + (b-2)(2a+2b-1)) \tilde{\Delta}^{-2} \partial_\alpha \tilde{\Delta} \partial_\beta \tilde{\Delta} \\
& + \tilde{g}_{\alpha\beta} [\frac{b}{2} \tilde{\Delta}^{-1} \tilde{\nabla}^2 \tilde{\Delta} + \frac{b}{4} (\kappa + 2a + 2b - 3) \tilde{\Delta}^{-2} \partial^\gamma \tilde{\Delta} \partial_\gamma \tilde{\Delta} + \frac{b}{2} \tilde{\Delta}^{-1} \mu_i^{-1} \partial^\gamma \mu_i \partial_\gamma \tilde{\Delta}] .
\end{aligned} \tag{C.6}$$

Note that $\tilde{R}_{\alpha\beta}$ as well as the covariant derivatives $\tilde{\nabla}_\alpha$ are defined with respect to the $k-1$ dimensional metric $d\tilde{s}^2 = \sum_{i=1}^k X_i^{-1} d\mu_i^2$. While these expressions are rather unwieldy, they simplify considerably in both the S^5 and the S^7 reductions, as many of the coefficients take on simple values.

Finally, by taking the trace of the above, we find the expression for the D -dimensional curvature scalar

$$\begin{aligned}
R = & \tilde{\Delta}^{-a} [R^{(d)} - (\kappa + a) \nabla^\rho (\tilde{\Delta}^{-1} \nabla_\rho \tilde{\Delta}) + \frac{1}{4} (\partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{g}^{\alpha\beta} - X_i^{-2} \partial^\rho X_i \partial_\rho X_i) \\
& + \frac{1}{4} (-(\kappa + a)(\kappa - 1) + 2b - (a+b)b(2k-1)) \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta}] \\
& + \tilde{\Delta}^b [\tilde{R} - (\kappa + 2a + b - 1) (\tilde{\Delta}^{-1} \tilde{\nabla}^2 \tilde{\Delta} + \tilde{\Delta}^{-1} \mu_i^{-1} \partial^\alpha \mu_i \partial_\alpha \tilde{\Delta}) \\
& + \frac{1}{4} (-(\kappa + 2a + b - 1)(\kappa + 2a + 2b - 5) - a(a+b)d) \tilde{\Delta}^{-2} \partial^\alpha \tilde{\Delta} \partial_\alpha \tilde{\Delta} \\
& - 2\mu_i^{-1} \tilde{\nabla}^2 \mu_i + \tilde{\Delta}^{-1} (\sum X)^2 - \tilde{\Delta}^{-1} \sum X^2] .
\end{aligned} \tag{C.7}$$

To make contact with the Kaluza-Klein reductions, we note that explicit computation of \tilde{R} yields

$$\tilde{R} = \tilde{\Delta}^{-1} [2\tilde{\Delta}^{-1} \sum X^3 \mu^2 - 2\tilde{\Delta}^{-1} \sum X \sum X^2 \mu^2 + (\sum X)^2 - \sum X^2] , \tag{C.8}$$

where we have followed a shorthand notation of removing indices so that, *e.g.* $\sum X^3 \mu^2 \equiv \sum_{i=1}^k X_i^3 \mu_i^2$. Note that for the special case of $k=3$, corresponding to S^5 , not all of the above quantities are independent. As a result we find that this expression simplifies to yield $\tilde{R}_{(k=3)} = 2\tilde{\Delta}^{-2} X_1 X_2 X_3 = 2\tilde{\Delta}^{-2}$. Additionally, we often find the following identities useful:

$$\begin{aligned}
\partial^\alpha \tilde{\Delta} \partial_\alpha \tilde{\Delta} &= -4 [\tilde{\Delta}^{-1} (\sum X^2 \mu^2)^2 - \sum X^3 \mu^2] , \\
\tilde{\nabla}^2 \tilde{\Delta} &= 2 [\tilde{\Delta}^{-2} (\sum X^2 \mu^2)^2 - \tilde{\Delta}^{-1} \sum X^3 \mu^2 - \tilde{\Delta}^{-1} \sum X \sum X^2 \mu^2 + \sum X^2] , \\
\mu_i^{-1} \partial^\alpha \mu_i \partial_\alpha \tilde{\Delta} &= -2 [\tilde{\Delta}^{-1} \sum X \sum X^2 \mu^2 - \sum X^2] , \\
\mu_i^{-1} \tilde{\nabla}^2 \mu_i &= \tilde{\Delta}^{-1} [\tilde{\Delta}^{-1} \sum X \sum X^2 \mu^2 - (\sum X)^2] .
\end{aligned} \tag{C.9}$$

The S^5 reduction of type IIB supergravity discussed in section 2 corresponds to the choice of $d = 5$ and $a = b = \frac{1}{2}$. In this case we obtain

$$\begin{aligned} \tilde{\Delta}^{1/2} R_{(k=3)}^{(D=10)} = & R^{(5)} - \frac{1}{2} \nabla^\rho (\tilde{\Delta}^{-1} \nabla_\rho \tilde{\Delta}) - \frac{1}{4} (-\partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{g}^{\alpha\beta} + X_i^{-2} \partial^\rho X_i \partial_\rho X_i + \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta}) \\ & + 2(\tilde{\Delta}^{-1} + 3 \sum X^{-1}) , \end{aligned} \quad (\text{C.10})$$

where we have used the simplified expression for $\tilde{R}_{(k=3)}$ given above. On the other hand, the S^7 reduction of eleven dimensional supergravity, given by the line element (3.1), corresponds to the choice of $d = 4$ and $a = \frac{2}{3}$, $b = \frac{1}{3}$. The eleven-dimensional curvature scalar is

$$\begin{aligned} \tilde{\Delta}^{2/3} R_{(k=4)}^{(D=11)} = & R^{(4)} - \frac{2}{3} \nabla^\rho (\tilde{\Delta}^{-1} \nabla_\rho \tilde{\Delta}) - \frac{1}{4} (-\partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{g}^{\alpha\beta} + X_i^{-2} \partial^\rho X_i \partial_\rho X_i + \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta}) \\ & - \frac{2}{3} \tilde{\Delta}^{-2} (\sum X^2 \mu^2)^2 + \frac{8}{3} \tilde{\Delta}^{-1} \sum X^3 \mu^2 - \frac{4}{3} \tilde{\Delta}^{-1} \sum X \sum X^2 \mu^2 \\ & + 4(\sum X)^2 - \frac{14}{3} \sum X^2 . \end{aligned} \quad (\text{C.11})$$

The last two lines involve undifferentiated scalars, and is used in (3.11). Curiously, the scalar kinetic terms in both cases have an identical structure save for a total derivative, and take on a standard Kaluza-Klein appearance (since $X_i^{-2} \partial^\rho X_i \partial_\rho X_i = -\partial^\rho \hat{g}_{ij} \partial_\rho \hat{g}^{ij}$). Finally note that the implicitly defined term $\partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{g}^{\alpha\beta}$ may be evaluated to give

$$-\partial^\rho \tilde{g}_{\alpha\beta} \partial_\rho \tilde{g}^{\alpha\beta} = X_i^{-2} \partial^\rho X_i \partial_\rho X_i + \tilde{\Delta}^{-2} \partial^\rho \tilde{\Delta} \partial_\rho \tilde{\Delta} - 2\tilde{\Delta}^{-1} X_i^{-1} \partial^\rho X_i \partial_\rho X_i \mu_i^2 . \quad (\text{C.12})$$

References

- [1] L.J. Romans, *Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory*, Nucl. Phys. **B383** (1992) 395; *Black holes in cosmological Einstein-Maxwell theory*, in “Trieste 1992, Proceedings, High energy physics and cosmology” 416.
- [2] K. Behrndt, A.H. Chamseddine and W.A. Sabra, *BPS black holes in $N = 2$ five-dimensional AdS supergravity*, Phys. Lett. **B442** (1998) 97, hep-th/9807187.
- [3] D. Birmingham, *Topological black holes in anti-de Sitter space*, hep-th/9808032.
- [4] M.M. Caldarelli and D. Klemm, *Supersymmetry of anti-de Sitter black holes*, hep-th/9808097.
- [5] D. Klemm, *BPS black holes in gauged $N = 4$ $D = 4$ supergravity*, hep-th/9810090.
- [6] K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes five-dimensional $N = 2$ AdS supergravity*, hep-th/9810227.

- [7] M.J. Duff and J.T. Liu, *Anti-de Sitter black holes in gauged $N=8$ supergravity*, hep-th/9901149.
- [8] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, hep-th/9902170.
- [9] M. Cvetič and S.S. Gubser, *Phases of R charged black holes, spinning branes and strongly coupled gauge theories*, hep-th/9902195.
- [10] M. Cvetič and S.S. Gubser, *Thermodynamic stability and phases of general spinning branes*, hep-th/9903132.
- [11] W. Sabra, *Anti-de Sitter black holes in $N=2$ gauged supergravity*, hep-th/9903143.
- [12] M.M. Caldarelli and D. Klemm, *M-theory and stringy corrections to anti-de Sitter black holes and conformal field theories*, hep-th/9903078.
- [13] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
- [14] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.
- [15] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.
- [16] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2** (1998) 505, hep-th/9803131.
- [17] B. de Wit and H. Nicolai, *$N = 8$ supergravity with local $SO(8) \times SU(8)$ invariance*, Phys. Lett. **B108** (1982) 285.
- [18] B. de Wit and H. Nicolai, *$N=8$ supergravity*, Nucl. Phys. **B208** (1982) 323.
- [19] M.J. Duff and C.N. Pope, *Kaluza-Klein supergravity and the seven sphere*, in: Super-symmetry and supergravity 82, eds. S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scientific, Singapore, 1983).
- [20] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. **130** (1986) 1.

- [21] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged maximally extended supergravity in seven dimensions*, Phys. Lett. **B143** (1984) 103.
- [22] M. Gunaydin, L.J. Romans and N. P. Warner, *Gauged $N = 8$ supergravity in five dimensions*, Phys. Lett. **B154** (1985) 268.
- [23] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, Nucl. Phys. **B226** (1983) 269.
- [24] M. Gunaydin and N. Marcus, *The spectrum of the S^5 compactification of the $N = 2$, $D = 10$ supergravity and the unitary supermultiplet*, Class. Quant. Grav. **2** (1985) L11.
- [25] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *Mass spectrum of chiral ten-dimensional $N = 2$ supergravity on S^5* , Phys. Rev. **D32** (1985) 389.
- [26] P.K. Townsend and P. van Nieuwenhuizen, *Gauged seven-dimensional supergravity*, Phys. Lett. **B125** (1983) 41-6.
- [27] K. Pilch, P. K. Townsend and P. van Nieuwenhuizen, *Compactification of $d = 11$ supergravity on S^4 (or $11 = 7 + 4$, too)*, Nucl. Phys. **B242** (1984) 377.
- [28] J.T. Liu and R. Minasian, *Black holes and membranes in AdS_7* , hep-th/9903269.
- [29] G.W. Gibbons and P.K. Townsend, *Vacuum interpolation in supergravity via super p -branes*, Phys. Rev. Lett. **71** (1993) 3754.
- [30] M.J. Duff, G.W. Gibbons and P.K. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. **B332** (1994) 321.
- [31] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, *Higher-dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. **12** (1995) 297.
- [32] M.J. Duff, *Anti-de Sitter space, branes, singletons, superconformal field theories and all that*, hep-th/9808100.
- [33] M.J. Duff and C.N. Pope, *Consistent truncations in Kaluza-Klein theories*, Nucl. Phys. **B255** (1985) 355.
- [34] C.N. Pope, *Consistency of truncations in Kaluza-Klein*, published in the Proceedings of the 1984 Santa Fe meeting.

- [35] B. de Wit, H. Nicolai and N.P. Warner, *The embedding of gauged $N = 8$ supergravity into $D = 11$ supergravity*, Nucl.Phys. **B255** (1985) 29.
- [36] B. de Wit and H. Nicolai, *The consistency of the S^7 truncation in $D = 11$ supergravity*, Nucl. Phys. **B281** (1987) 211.
- [37] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, Nucl. Phys. **B459** (1996) 125, hep-th/9508094.
- [38] M. Cvetič and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, Phys. Rev. **D53** (1996) 584, hep-th/9507090.
- [39] H. Lü and C.N. Pope, *p -brane solitons in maximal supergravities*, Nucl. Phys. **B465** (1996) 127, hep-th/9512012.
- [40] R.R. Khuri and T. Ortin, *Supersymmetric black holes in $N = 8$ supergravity*, Nucl. Phys. **B467** (1996) 355.
- [41] M. Cvetič and A.A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, Phys. Lett. **B366** (1996) 95, hep-th/9510097.
- [42] A.A. Tseytlin, *Harmonic superpositions of M-branes*, Nucl. Phys. **B475** (1996) 149, hep-th/9604035.
- [43] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, *Compactifications of the eleven-dimensional supermembrane*, Phys. Lett. **B224**, 71 (1989).
- [44] E. Bergshoeff, E. Sezgin and P. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **B189** (1987) 75.
- [45] M.J. Duff and K.S. Stelle, *Multi-membrane solutions of $D = 11$ supergravity*, Phys. Lett. **B253** (1991) 113.
- [46] M. Cvetič and D. Youm, *Near BPS saturated rotating electrically charged black holes as string states*, Nucl. Phys. **B477** (1996) 449, hep-th/9605051.
- [47] M. Cvetič and D. Youm, *Rotating intersecting M-branes*, Nucl. Phys. **B499** (1997) 253, hep-th/9612229.
- [48] S. Gubser, *Thermodynamics of spinning D3-branes*, hep-th/9810225.

- [49] C. Csaki, Y. Oz, J. Russo and J. Terning, *Large N QCD from rotating branes*, hep-th/9810186.
- [50] P. Kraus, F. Larsen and S.P. Trivedi, *The Coulomb branch of gauge theory from rotating branes*, hep-th/9901056.
- [51] R. Cai and K. Soh, *Critical behavior in the rotating D-branes*, hep-th/9812121.
- [52] P. Kraus, F. Larsen and S.P. Trivedi, *The Coulomb branch of gauge theory from rotating branes*, hep-th/9811120.
- [53] J.G. Russo and K. Sfetsos, *Rotating D3-branes and QCD in three dimensions*, hep-th/9901056.
- [54] C. Csaki, J. Russo, K. Sfetsos and J. Terning, *Supergravity Models for 3+1 Dimensional QCD*, hep-th/9902067.
- [55] K. Sfetsos, *Rotating NS5-brane solution and its exact string theoretical description*, hep-th/9903201.
- [56] K.S. Thorne, *Black holes: the membrane paradigm*, New Haven, USA: Yale Univ. Pr. (1986) 367p.
- [57] R.C. Myers and M.J. Perry, *Black holes in higher dimensional spacetimes*, Ann. Phys. **172** (1986) 304.
- [58] M. Cvetič, H. Lü and C.N. Pope, *Spacetimes of boosted p-branes and CFT in infinite momentum frame*, hep-th/9810123, to appear in Nucl. Phys. **B**.
- [59] B.E.W. Nilsson, *On the embedding of $d = 4$, $N = 8$ gauged supergravity in $d = 11$, $N = 1$ supergravity*, Phys. Lett. **B155** (1985) 54.
- [60] C.N. Pope, *The embedding of the Einstein-Yang-Mills equations in $D = 11$ supergravity*, Class. Quantum Grav. **2** (1985) L77.
- [61] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Noncompact gaugings and critical points of maximal supergravity in seven dimensions*, Nucl. Phys. **B249** (1985) 381.
- [62] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Dilatonic p-brane solitons*, Phys. Lett. **B371** (1996) 46, hep-th/9511203.

- [63] M.J. Duff and J.X. Lu, *Black and super p -branes in diverse dimensions*, Nucl. Phys. **B416** (1994) 301, hep-th/9306052.
- [64] E. Cremmer and B. Julia, *The $N = 8$ supergravity theory. 1. The Lagrangian*, Phys. Lett. **B80** (1978) 48.
- [65] E. Cremmer and B. Julia, *The $SO(8)$ supergravity*, Nucl. Phys. **B159** (1979) 141.
- [66] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities*, Nucl. Phys. **B523** (1998) 73, hep-th/9710119.